

EXPLICIT REALIZATION OF HOPF CYCLIC COHOMOLOGY CLASSES OF BICROSSED PRODUCT HOPF ALGEBRAS

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ABSTRACT. We construct a Hopf action, with an invariant trace, of a bicrossed product Hopf algebra $\mathcal{H} = (\mathcal{U}(\mathfrak{g}_1) \bowtie \mathcal{R}(G_2))^{\text{cop}}$ constructed from a matched pair of Lie groups G_1 and G_2 , on a convolution algebra $\mathcal{A} = C_c^\infty(G_1) \rtimes G_2^\delta$. We give an explicit way to construct Hopf cyclic cohomology classes of our Hopf algebra \mathcal{H} and then realize these classes in terms of explicit representative cocycles in the cyclic cohomology of the convolution algebra \mathcal{A} .

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1. INTRODUCTION

The geometric Hopf algebras and their cyclic cohomology first appeared as a new geometric tool in the work [7] of Connes and Moscovici on the local index formula for transversely hypoelliptic operators on foliations. They constructed a canonical isomorphism [7, Thm. 11] from the Gelfand-Fuks cohomology of the Lie algebra of formal vector fields on \mathbb{R}_n ([13]) to the Hopf cyclic cohomology of the Hopf algebra $\mathcal{H}(n)$ associated to the pseudogroup of local diffeomorphisms of \mathbb{R}_n . In this way they proved that the local index formula computes characteristic classes of foliations ([1]). A detailed account of the above isomorphism, based on the bicrossed product decomposition of $\mathcal{H}(n)$, was provided by Moscovici and Rangipour ([32]), see the diagram from [32, 3.61]:

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$$\begin{array}{ccccc}
& & \mathcal{D} & & \\
& \swarrow & & \searrow & \\
C_{\text{top}}^{\bullet}(\mathfrak{a}_n) & \xrightarrow{\mathcal{E}} & C_{c-w}^{\bullet,\bullet}(\mathfrak{g}^*, \mathcal{F}) & \xrightarrow{\Theta} & C_{\text{Bott}}^{\bullet}(\Omega_{\bullet}(G), \Gamma) \\
& & \uparrow & & \downarrow \Phi \\
& & C^{\bullet}(\mathcal{H}(n), \mathbb{C}_{\delta}) & & C^{\bullet}(C_c^{\infty}(G) \rtimes \Gamma)
\end{array}$$

where $G = \mathbb{R}^n \rtimes GL(n)$; Γ is a subgroup of $\text{Diff}(n)$, diffeomorphisms group of \mathbb{R}^n .

Recently, Moscovici gave an explicit description of a basis of Hopf cyclic characteristic classes of $\mathcal{H}(n)$ ([29]) in the spirit of Chern-Weil theory. He showed that the image of \mathcal{D} forms a subcomplex of the Bott complex $C_{\text{Bott}}^{\bullet}(\Omega_{\bullet}(G), \Gamma)$ from diagram above, which is quasi-isomorphic to Hopf cyclic via a restriction of Connes' Φ map. That is

$$\begin{array}{ccccccc}
& & \mathcal{D} & & & & \\
& \swarrow & & \searrow & & & \\
C_{\text{top}}^{\bullet}(\mathfrak{a}_n) & \xrightarrow{\mathcal{E}} & C_{c-w}^{\bullet,\bullet}(\mathfrak{g}^*, \mathcal{F}) & \xrightarrow{\Theta} & \text{Im}(\mathcal{D}) & & \\
& & & & \downarrow \Phi_d & & \\
C^{\bullet}(\mathcal{H}(n), \mathbb{C}_{\delta}) & \xrightarrow[\cong]{\lambda} & \text{Im}(\lambda) & \hookrightarrow & C^{\bullet}(C_c^{\infty}(G) \rtimes \text{Diff}(n)) & \longrightarrow & C^{\bullet}(C_c^{\infty}(G) \rtimes \Gamma)
\end{array}$$

This completes the description of relationship between the Lie algebra cohomology of \mathfrak{a}_n , the Hopf cyclic cohomology of the Hopf algebra $\mathcal{H}(n)$ and the cyclic cohomology of $C_c^{\infty}(G) \rtimes \text{Diff}(n)$ and $C_c^{\infty}(G) \rtimes \Gamma$. Note that the cohomology classes of the Lie algebra of formal vector fields are sophisticated, the calculation above is carried out by \mathcal{D} map using simplicial connection, instead of calculating \mathcal{E} and Θ explicitly.

In this paper we are implementing a similar procedure for a bicrossed product Hopf algebra associated to a matched pair of Lie groups. For a matched pair of Lie groups G_1, G_2 , with corresponding bicrossed product Lie algebra $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$, one can also construct a bicrossed product Hopf algebra $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$. Rangipour and S  tl   ([34]) showed that the Hopf cyclic cohomology of the Hopf algebra $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$ is canonically isomorphic via a van Est type quasi-isomorphism to the Lie algebra cohomology of $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ relative to a certain Lie subalgebra \mathfrak{h}_2 , as in the following diagram (both arrows are quasi-isomorphisms):

$$\begin{array}{ccc}
C(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{h}_2) & \xleftarrow{\nu} & C^{\bullet,\bullet}(\mathfrak{g}_1^*, \mathcal{R}_2) \\
& & \uparrow \mathcal{J} \\
& & C^{\bullet}(\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1), {}^{\sigma^{-1}}\mathbb{C}_{\delta})
\end{array}
\tag{1.1}$$

In their work, a quasi-isomorphism from the complex $C^{\bullet}(\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1), {}^{\sigma^{-1}}\mathbb{C}_{\delta})$ of $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$ with coefficient in some module comodule ${}^{\sigma^{-1}}\mathbb{C}_{\delta}$ to the Lie algebra cohomology complex of $(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{h}_2, M)$ is constructed. However, in order to write down the

Hopf cyclic classes explicitly, one would need the precise expression of the map from Lie algebra cohomology complex to the cyclic complex of Hopf algebra.

We invert the map ν , following the integration along simplices in G/K map from the Lie algebra complex to the continuous group complex as in [12] and [39].

Instead of inverting \mathcal{J} , we find a Hopf action (3.19), with an invariant trace, of it on the convolution algebra $\mathcal{A} = C_c^\infty(G_1) \rtimes G_2^\delta$ (δ for the discrete topology), as well as the convolution algebra $\mathcal{A}_\Gamma = C_c^\infty(G_1) \rtimes \Gamma$ for any discrete subgroup Γ of G_2 . Next we use Connes' Φ map to go directly to the cyclic cohomology of the algebra. In this case the image of Φ sits inside the range of the characteristic map from the Hopf cyclic cohomology.

Compared with the work of Moscovici's ([29]), unlike the Lie algebra of formal vector fields whose cohomology classes are sophisticated, we take the advantage that for finite dimensional Lie algebra one can represent the cohomology classes by invariant forms explicitly. This way we can use \mathcal{E} and Θ directly and write down the explicit Hopf cyclic classes of our Hopf algebra $\mathcal{H} = (\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1))^{\text{cop}}$ in terms of representative cocycles on the convolution algebra, as illustrated by the following diagram:

$$\begin{array}{ccccc}
 & & \mathcal{D} & & \\
 & \swarrow & \text{---} & \searrow & \\
 C(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{h}_2) & \xrightarrow{\mathcal{E}} & C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*) & \xrightarrow{\Theta} & D^{p,q} \\
 & & & \downarrow \Phi_D & \\
 & & C^\bullet(\mathcal{H}, {}^\sigma \mathbb{C}_\delta) & \xrightarrow[\cong]{\lambda} & \text{Im}(\lambda) \hookrightarrow C^\bullet(\mathcal{A}) \longrightarrow C^\bullet(\mathcal{A}_\Gamma)
 \end{array}
 \tag{1.2}$$

This amounts to an indirect way of inverting maps in diagram 1.1, and be used to give a explicit description of Hopf cyclic classes of $\mathcal{H} = (\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1))^{\text{cop}}$. Let us known that the Hopf algebra $(\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1))^{\text{cop}}$ actually acts on $\mathcal{A}_\Gamma = C_c^\infty(G_1) \rtimes \Gamma$ as well for any discrete subgroup $\Gamma \subset G_2$, so we can complete the diagram with the very bottom right corner.

The paper is organized as follows. In section 2 we introduce some background material. In section 3 we construct the bicrossed product Hopf algebra $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$. We show that the induced morphism on (Theorem 3.19) Hopf action of $(\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1))^{\text{cop}}$ on a convolution algebra $\mathcal{A} = C_c^\infty(G_1) \rtimes G_2^\delta$. We introduce differentiable and representative cohomologies of action groupoid and the complexes that calculate them in section 4. We also explain the construction of $D^{p,q}$ as a smaller subcomplex of the previous complexes. At the end of section 4 we give an account of Connes' Φ map in section 4, whose restriction Φ_D on $D^{p,q}$ is going to be the quasi-isomorphism we need.

At the beginning of section 5 we review the maps appear in the work of Rangipour and Sütülü ([34]) as listed in diagram 1.1 under our setting ($\sigma^{-1}M_\delta = \sigma^{-1}\mathbb{C}_\delta$). Then we describe the maps \mathcal{E} , \mathcal{D} , Θ in the top row of the diagram above and show that all of them are quasi-isomorphisms in the second section. At the end of section 5 we connect the maps from section 3 to section 5 and prove theorem 5.11 that gives the same result of Rangipour and Sütülü ([34, Thm. 4.10, Cor. 4.11]) but from a different direction with explicit formulas.

Finally we give an example calculation of transition from Lie algebra cohomology classes as explicit invariant forms on a 4 dimensional Lie group to cyclic cochains on the corresponding convolution algebra \mathcal{A} in [section 6](#).

2. PRELIMINARIES AND NOTATIONS

In this chapter we provide background material that will be needed. We first recall the definitions of matched pair of Lie groups and Lie algebras, and then the bicrossed product Hopf algebra constructions. Most of the material is taken from [\[22, 28, 6\]](#).

2.1. Matched pair of Lie groups and Lie algebras. First, we would like to define a matched pair of Lie groups and a matched pair of Lie algebras. The definitions are given in Takeuchi's paper [\[38\]](#), Majid's paper [\[27\]](#) and book [\[28\]](#):

Definition 2.1. [\[38, 27, 28\]](#) *Two Lie groups (G_1, G_2) are a matched pair if they act on each other and the left action \triangleright of G_2 on G_1 , the right action \triangleleft of G_1 on G_2 , obey the conditions:*

$$(2.1) \quad \begin{aligned} \forall \psi_1, \psi_2 \in G_2, \quad \varphi_1, \varphi_2 \in G_1, \quad \psi_1 \triangleright e = e, \quad e \triangleleft \varphi_1 = e, \\ \psi_1 \triangleright (\varphi_1 \varphi_2) = (\psi_1 \triangleright \varphi_1)((\psi_1 \triangleleft \varphi_1) \triangleright \varphi_2), \\ (\psi_1 \psi_2) \triangleleft \varphi_1 = (\psi_1 \triangleleft (\psi_2 \triangleright \varphi_1))(\psi_2 \triangleleft \varphi_1). \end{aligned}$$

One example would be a matched pair of Lie subgroups from a decomposition of a Lie group. Let $G = G_1 G_2$ as sets, $G_1 \cap G_2 = e$, one defines the actions to be:

$$(2.2) \quad \psi \varphi = (\psi \triangleright \varphi)(\psi \triangleleft \varphi), \quad \text{for } \psi \in G_2 \text{ and } \varphi \in G_1,$$

and check that \triangleright and \triangleleft are matched actions.

Differentiate these Lie groups we get a infinitesimal version: matched pair of Lie algebras:

Definition 2.2. [\[27, 28\]](#) *Two Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$ are a matched pair if they act on each other and the left action \triangleright of \mathfrak{g}_2 on \mathfrak{g}_1 , the right action \triangleleft of \mathfrak{g}_1 on \mathfrak{g}_2 , obey the conditions:*

$$(2.3) \quad \begin{aligned} [X_1, X_2] \triangleright Y &= X_1 \triangleright (X_2 \triangleright Y) - X_2 \triangleright (X_1 \triangleright Y), \\ X \triangleleft [Y_1, Y_2] &= (X \triangleleft Y_1) \triangleleft Y_2 - (X \triangleleft Y_2) \triangleleft Y_1, \\ X \triangleright [Y_1, Y_2] &= [X \triangleleft Y_1, Y_2] + [Y_1, X \triangleright Y_2] + (X \triangleleft Y_1) \triangleright Y_2 - (X \triangleleft Y_2) \triangleright Y_1, \\ [X_1, X_2] \triangleleft Y &= [X_1 \triangleleft Y, X_2] + [X_1, X_2 \triangleleft Y] + X_1 \triangleleft (X_2 \triangleright Y) - X_2 \triangleleft (X_1 \triangleright Y). \end{aligned}$$

Similar to the group decomposition example, one would get a matched pair of Lie subalgebras from a decomposition of a Lie algebra. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as vector spaces, one defines the actions to be:

$$(2.4) \quad [Y, X] = Y \triangleright X + Y \triangleleft X, \quad \text{for } Y \in \mathfrak{g}_2 \text{ and } X \in \mathfrak{g}_1,$$

and check that \triangleright and \triangleleft are matched actions.

One can also assemble a match pair of Lie groups or Lie algebras to one Lie group or Lie algebra. We will introduce this when needed (cf. [4.1](#)).

2.2. Bicrossed product Hopf algebras. We refer the complete argument to [28]. We will use Sweedler notation [37] to denote a comultiplication by $\Delta(c) = c_{(1)} \otimes c_{(2)}$, omitting the summation.

Definition 2.3. Let C be a coalgebra. A left C -comodule is a pair (N, Δ_N) where N is a vector space and $\Delta_N : N \rightarrow C \otimes N$ is a linear map, called the coaction of C on N , such that the following diagrams commute:

$$\begin{array}{ccc} N & \xrightarrow{\Delta_N} & C \otimes N \\ \downarrow \Delta_N & & \downarrow Id \otimes \Delta_N \\ C \otimes N & \xrightarrow{\Delta \otimes Id} & C \otimes C \otimes N \end{array} \quad \begin{array}{ccc} C \otimes N & \xleftarrow{\varepsilon \otimes Id} & C \otimes N \\ & \nwarrow \cong & \uparrow \Delta_N \\ & & N \end{array}$$

We would also use Sweedler notation to denote a left coaction by $\Delta_N(c) = c_{<-1>} \otimes c_{<0>}$. Similarly we will denote right coaction of a coalgebra on a comodule N by $\Delta_N(c) = c_{<0>} \otimes c_{<1>}$. One shall take advantage of the signs in this notation as $c_{<0>}$ always stay in the same vector space N while $c_{<-1>}, c_{<1>}$ are in the coalgebra C , $-$, $+$ signs tell the left/right direction of the coaction.

More detailed information on coalgebras, Hopf algebras and their comodules can be found in [22]

Definition 2.4. When a Hopf algebra \mathcal{H} acts on an algebra A from the left, we say that A is a left \mathcal{H} -module algebra if

$$\begin{aligned} h \triangleright (ab) &= (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \\ h \triangleright (1_A) &= \varepsilon(h)1_A. \end{aligned}$$

Then we can form a left cross product algebra $A \rtimes \mathcal{H}$ built on vector space $A \otimes \mathcal{H}$ with the product:

$$\begin{aligned} (a \rtimes h)(b \rtimes g) &= a(h_{(1)} \triangleright b) \otimes h_{(2)}g, \quad a, b \in A, \quad h, g \in \mathcal{H} \\ 1_{A \rtimes \mathcal{H}} &= 1_A \otimes 1_{\mathcal{H}}. \end{aligned}$$

Definition 2.5. Let \mathcal{H} be a Hopf algebra. Suppose a coalgebra C is a right \mathcal{H} -comodule with coaction

$$\blacktriangledown(c) = c_{<0>} \otimes c_{<1>},$$

then C is a right \mathcal{H} -comodule coalgebra if the coaction commutes with the coproduct and counit, i.e., the following conditions are satisfied:

$$\begin{aligned} c_{<0>{(1)}} \otimes c_{<0>{(2)}} \otimes c_{<1>} &= c_{(1)<0>} \otimes c_{(2)<0>} \otimes c_{(1)<1>}c_{(2)<1>} \\ \varepsilon(c_{<0>})c_{<1>} &= \varepsilon(c)1_{\mathcal{H}}. \end{aligned}$$

Then we can form a right cross coproduct coalgebra $\mathcal{H} \bowtie C$ built on vector space $\mathcal{H} \otimes C$ with the coalgebra structure:

$$\begin{aligned} \Delta(h \bowtie c) &= h_{(1)} \bowtie c_{(1)<0>} \otimes h_{(2)}c_{(1)<1>} \bowtie c_{(2)}, \\ \varepsilon(h \bowtie c) &= \varepsilon(h)\varepsilon(c), \end{aligned}$$

for $h \in \mathcal{H}$ and $c \in C$.

Definition 2.6. Let A, \mathcal{H} be Hopf algebras, let A be a left \mathcal{H} -module algebra and \mathcal{H} be a right A -comodule coalgebra such that the compatibility conditions

$$\begin{aligned} \varepsilon(h \triangleright a) &= \varepsilon(h)\varepsilon(a), \quad \Delta(h \triangleright a) = h_{(1)<0>} \triangleright a_{(1)} \otimes h_{(1)<1>} (h_{(2)} \triangleright a_{(2)}), \\ 1_{<0>} \otimes 1_{<1>} &= 1 \otimes 1, \\ (gh)_{<0>} \otimes (gh)_{<1>} &= g_{(1)<0>} h_{<0>} \otimes g_{(1)<1>} (g_{(2)} \triangleright h_{<1>}), \\ h_{(2)<0>} \otimes (h_{(1)} \triangleright a) h_{(2)<1>} &= h_{(1)<0>} \otimes h_{(1)<1>} (h_{(2)} \triangleright a), \end{aligned}$$

are satisfied for all $a, b \in A$ and $g, h \in \mathcal{H}$. Then the algebra $A \bowtie \mathcal{H}$ and coalgebra $A \blacktriangleright \mathcal{H}$ form a left-right bicrossed product Hopf algebra $A \blacktriangleright \blacktriangleleft \mathcal{H}$ with antipode

$$S_{\blacktriangleright \blacktriangleleft}(a \blacktriangleright h) = (1 \blacktriangleright S(h_{<0>}))(S(ah_{<1>}) \blacktriangleleft 1)$$

2.3. Additional notation. In this paper G will denote a Lie group, G^δ means same group but with discrete topology, H and L are closed subgroups; $\mathfrak{g}, \mathfrak{h}$, and \mathfrak{l} will denote the corresponding Lie algebras. X, Y, Z are vectors of Lie algebras. $\tilde{X}, \tilde{Y}, \tilde{Z}$ means the corresponding left invariant vector field. \mathcal{H} is Hopf algebra, \mathcal{A} is convolution algebra. \blacktriangledown is for coaction and $\blacktriangleright \blacktriangleleft$ is for left-right bicrossed product. \mathcal{U} is universal enveloping algebra, \mathcal{R} is algebra of representative functions.

3. HOPF ALGEBRA \mathcal{H} AND ITS STANDARD ACTION

3.1. Hopf algebra \mathcal{H} . In this section, our goal is to associate a bicrossed product Hopf algebra and a convolution algebra on which the Hopf algebra acts to any matched pair of Lie groups. We prove theorem 3.19 and show that the characteristic map is faithful.

Given a matched pair of Lie groups (G_1, G_2) , we view G_2 as discrete group and denote as G_2^δ . Consider the action groupoid $\mathcal{G} = G_1 \rtimes G_2^\delta$ and its convolution algebra $C_c^\infty(\mathcal{G})$, which is equivalent to the cross product algebra $\mathcal{A} = C_c^\infty(G_1) \rtimes G_2^\delta$ (see [35, 10]). The elements of \mathcal{A} are finite sums of symbols of the form

$$fU_\psi^*, \quad \text{where } f \in C_c^\infty(G_1), \quad U_\psi^* \text{ stands for } \widetilde{\psi^{-1}} \in G_2,$$

with multiplication:

$$(3.1) \quad fU_{\psi_1}^* * gU_{\psi_2}^* = f(g \circ \widetilde{\psi_1})U_{\psi_2\psi_1}^*,$$

where $\widetilde{\psi_1} \in \text{Diff}(G_1)$ is left action by ψ_1 . We will also use $fU_\psi := fU_{\psi^{-1}}^*$ to simplify notation afterwards.

Denote the Lie algebra of G_1 by \mathfrak{g}_1 and its basis element by $Z_i, 1 \leq i \leq \dim(G_1)$, the corresponding left-invariant vector field on G_1 by \tilde{Z}_i . These vector fields act on \mathcal{A} by

$$(3.2) \quad \tilde{Z}_i(fU_\psi^*) = \tilde{Z}_i(f)U_\psi^*, \quad \tilde{Z}_i(f)U_\psi^*|_{\varphi_0} = \frac{d}{dt} \Big|_{t=0} f(\varphi_0 \exp(tZ_i))U_\psi^*.$$

Now we calculate $L_{\psi^*} \tilde{Z}_i$ for $\psi \in G_2$:

$$\begin{aligned}
 \tilde{Z}_i(f \circ \tilde{\psi})|_{\varphi_0} &= \frac{d}{dt} \Big|_{t=0} (f \circ \tilde{\psi})(\varphi_0 \exp(tZ_i)) \\
 &= \frac{d}{dt} \Big|_{t=0} f\left(\psi \triangleright (\varphi_0 \exp(tZ_i))\right) \\
 (3.3) \quad &= \frac{d}{dt} \Big|_{t=0} f\left((\psi \triangleright \varphi_0)((\psi \triangleleft \varphi_0) \triangleright \exp(tZ_i))\right) \\
 &= \frac{d}{dt} \Big|_{t=0} f\left((\psi \triangleright \varphi_0)((\psi^{-1} \triangleleft (\psi \triangleright \varphi_0))^{-1} \triangleright \exp(tZ_i))\right) \\
 &= \sum_j (\Gamma_i^j(\psi^{-1}) \tilde{Z}_j(f)) \circ \tilde{\psi}|_{\varphi_0},
 \end{aligned}$$

or in short

$$(3.4) \quad \tilde{Z}_i U_{\psi}^* = \sum_j U_{\psi}^* \Gamma_i^j(\psi^{-1}) \tilde{Z}_j,$$

where

$$(3.5) \quad \sum_j \Gamma_i^j(\psi)(\varphi_0) Z_j := (\psi \triangleleft \varphi_0)^{-1} \triangleright Z_i, \quad \text{or} \quad \Gamma_i^j(\psi)(\varphi_0) := \langle (\psi \triangleleft \varphi_0)^{-1} \triangleright Z_i, \omega_j \rangle,$$

and $\{\omega_j, 1 \leq j \leq \dim(G_1)\}$ is the dual basis and the action \triangleright of G_2 on \mathfrak{g}_1 is given by the differentiation of \triangleright on G_1 at e .

We notice that Γ satisfies:

$$\begin{aligned}
 (3.6) \quad \Gamma_i^j(\psi_1 \psi_2) &= \sum_k (\Gamma_i^k(\psi_1) \circ \tilde{\psi}_2) \Gamma_k^j(\psi_2), \\
 \Gamma_i^j(\psi^{-1}) \circ \tilde{\psi} &= (\Gamma^{-1})_i^j(\psi).
 \end{aligned}$$

From 3.5 the definition of Γ_i^j , we can see that Γ_i^j is determined only by its value at e :

$$(3.7) \quad \Gamma_i^j(\psi)(\varphi_0) = \Gamma_i^j(\psi \triangleleft \varphi_0)(e),$$

we will denote

$$(3.8) \quad \gamma_i^j(\psi) := \Gamma_i^j(\psi)(e), \quad \text{or} \quad \gamma_i^j(\psi) := \langle \psi^{-1} \triangleright Z_i, \omega_j \rangle.$$

Hence similarly we can show γ is a homomorphism:

$$\begin{aligned}
 (3.9) \quad \gamma_i^j(\psi_1 \psi_2) &= \gamma_i^k(\psi_1) \gamma_k^j(\psi_2), \\
 \gamma_i^j(\psi^{-1}) &= (\gamma^{-1})_i^j(\psi)
 \end{aligned}$$

This property of γ_i^j suggests that γ_i^j falls into a special kind of functions on G_2 called representative functions. We will give the definition originated from [20].

Definition 3.1. [20] Fix the base field to be \mathbb{C} , let G be a Lie group and $\rho : G \rightarrow \text{Aut}(V)$ a finite dimensional smooth representation. Topologize $\text{End}(V)$ so that every linear functional is continuous and topologize $\text{Aut}(V)$ by the induced topology. Then, the composition of ρ with a linear functional $\tau \in \text{Aut}(V)^*$ is called a representative function of G . Denote $\mathcal{R}(G)$ the algebra of representative functions when we run over all pairs of finite dimensional representation and linear functional.

It is well known that the representative functions $\mathcal{R}(G)$ form a commutative Hopf algebra:

$$(3.10) \quad \begin{aligned} F_1 F_2(\psi) &= F_1(\psi) F_2(\psi), \quad \Delta(F)(\psi_1, \psi_2) = F(\psi_1 \psi_2), \\ 1(\psi) &= 1, \quad \varepsilon(F) = F(e), \quad S(F)(\psi) = F(\psi^{-1}). \end{aligned}$$

From 3.4 we can uniquely express

$$(3.11) \quad U_\psi^* \tilde{Z}_i U_\psi = \Gamma_i^j(\psi) \tilde{Z}_j, \quad \text{or} \quad \widetilde{(\psi^{-1})_* (\tilde{Z}_i)} = \Gamma_i^j(\psi) \tilde{Z}_j.$$

This gives a left action of G_2 on \mathfrak{g}_1 :

$$(3.12) \quad \psi \triangleright (Z_i) = \widetilde{(\psi)_* (\tilde{Z}_i)}|_e = \Gamma_i^j(\psi^{-1})(e) Z_j = \gamma_i^j(\psi^{-1}) Z_j = S(\gamma_i^j)(\psi) Z_j,$$

which can be dualized using the identification

$$(3.13) \quad (Z_j)_{<0>} (Z_j)_{<1>}(\psi) := \psi \triangleright (Z_i)$$

to a right coaction of $\mathcal{R}(G_2)$ on \mathfrak{g}_1 :

$$(3.14) \quad \nabla(Z_i) = (Z_i)_{<0>} \otimes (Z_i)_{<1>} := Z_j \otimes S(\gamma_i^j).$$

The transpose of the left action 3.12 gives a right action of G_2 on \mathfrak{g}_1^* :

$$(3.15) \quad (\omega_i) \triangleright^t \psi = S(\gamma_i^j)(\psi) \omega_j,$$

dualize it we can have a left coaction of $\mathcal{R}(G_2)$ on \mathfrak{g}_1^* :

$$(3.16) \quad \nabla(\omega_i) := S(\gamma_i^j) \otimes \omega_j.$$

We also have a left action of \mathfrak{g}_1 on $\mathcal{R}(G_2)$ from differentiating the left action of G_1 on $C^\infty(G_2)$ (which comes from the right action of G_1 on G_2):

$$(3.17) \quad (Z_i \triangleright F)(\psi) := \left. \frac{d}{dt} \right|_{t=0} F(\psi \triangleleft \exp(tZ_i)),$$

Proposition 3.2. [33, 2.14] *For any $Z_i \in \mathfrak{g}_1$ and any $F \in \mathcal{R}(G_2)$, we have $(Z_i \triangleright F) \in \mathcal{R}(G_2)$.*

Both coaction and action can be extended from \mathfrak{g}_1 to $\mathcal{U}(\mathfrak{g}_1)$ and we have the following result.

Proposition 3.3. [33, Theorem 2.17] *by the left action 3.17 and the right coaction 3.14, the pair $(\mathcal{U}(\mathfrak{g}_1), \mathcal{R}(G_2))$ is a matched pair of Hopf algebras. i.e., $\mathcal{R}(G_2)$ is a left $\mathcal{U}(\mathfrak{g}_1)$ -module algebra, $\mathcal{U}(\mathfrak{g}_1)$ is a right $\mathcal{R}(G_2)$ -comodule coalgebra, and they satisfy the compatibility conditions*

$$\begin{aligned} \varepsilon(u \triangleright F) &= \varepsilon(u) \varepsilon(F), \quad \nabla(1) = 1 \otimes 1, \\ \Delta(u \triangleright F) &= u_{(1)<0>} \triangleright F_{(1)} \otimes u_{(1)<1>} (u_{(2)} \triangleright F_{(2)}), \\ \nabla(uv) &= u_{(1)<0>} v_{<0>} \otimes u_{(1)<1>} (u_{(2)} \triangleright v_{<1>}), \\ u_{(2)<0>} \otimes (u_{(1)} \triangleright F) u_{(2)<1>} &= u_{(1)<0>} \otimes u_{(1)<1>} (u_{(2)} \triangleright F). \end{aligned}$$

As a result of this proposition, for a matched pair of Hopf algebras $(\mathcal{U}(\mathfrak{g}_1), \mathcal{R}(G_2))$ we can form the left-right bicrossed product Hopf algebra $\mathcal{R}(G_2) \blacktriangleright \blacktriangleleft \mathcal{U}(\mathfrak{g}_1)$ in a canonical way.

The algebra structure is given by $\mathcal{R}(G_2) \blacktriangleright \blacktriangleleft \mathcal{U}(\mathfrak{g}_1)$:

$$(3.18) \quad (F \blacktriangleright u)(G \blacktriangleright v) = (F(u_{(1)} \triangleright G) \blacktriangleright u_{(2)} v)$$

with unit $(1 \blacktriangleright 1)$ and the coalgebra structure is given by $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$:

$$(3.19) \quad \Delta_{\blacktriangleright}(F \blacktriangleright u) = (F_{(1)} \blacktriangleright u_{(1)_{<0>}}) \otimes (F_{(2)} u_{(1)_{<1>}} \blacktriangleright u_{(2)})$$

with counit $\varepsilon(F \blacktriangleright u) = \varepsilon(F)\varepsilon(u)$, and finally the antipode is given by

$$(3.20) \quad S_{\blacktriangleright}(F \blacktriangleright u) = (1 \blacktriangleright S(u_{<0>}))(S(F u_{<1>}) \blacktriangleright 1)$$

Fix an order of the basis of \mathfrak{g}_1 and Let $I = (i_1, \dots, i_p)$, $p = \dim G_1$ be multi-indices, ordered lexicographically. We then have $Z_I = Z_1^{i_1} \dots Z_p^{i_p}$ as a PBW basis of $\mathcal{U}(\mathfrak{g}_1)$. Every element $\sum_j F_j \blacktriangleright u_j$ in $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$ can be written uniquely as

$$(3.21) \quad \sum_j F_j \blacktriangleright u_j = \sum_I F_I \blacktriangleright Z_I, \quad I = (i_1, \dots, i_p).$$

Therefore, $\{F \blacktriangleright Z_i, F \in \mathcal{R}(G_2), 1 \leq i \leq \dim G_1\}$ is a generating set of $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$.

Proposition 3.4. *For the specific Hopf algebra $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$ we can show that S_{\blacktriangleright} is invertible and $S_{\blacktriangleright}^{-1}$ is given by the formula*

$$(3.22) \quad S_{\blacktriangleright}^{-1}(F \blacktriangleright Z) = (S(Z_{<1>}) \blacktriangleright S(Z_{<0>}))(S(F) \blacktriangleright 1)$$

on the generators and is extended as an anti-algebra morphism.

Proof. The non-trivial check will be on $1 \blacktriangleright Z_i$: First we observe that $S(Z) = -Z$ for $Z \in \mathfrak{g}_1$ and

$$(3.23) \quad (-Z)_{<0>} \otimes (-Z)_{<1>} = \nabla(-Z) = -\nabla(Z) = -(Z)_{<0>} \otimes (Z)_{<1>}, \quad Z \in \mathfrak{g}_1$$

therefore

$$(3.24) \quad (S(Z))_{<0>} \otimes (S(Z))_{<1>} = S((Z)_{<0>}) \otimes (Z)_{<1>}, \quad Z \in \mathfrak{g}_1$$

note that the above equality is only true for $Z \in \mathfrak{g}_1$ not $u \in \mathcal{U}(\mathfrak{g}_1)$. The above equality allow us to interchange coaction and antipode on generators.

Now we have

$$\begin{aligned}
 S_{\blacktriangleleft}^{-1} S_{\blacktriangleleft}(1 \blacktriangleright Z) &= S_{\blacktriangleleft}^{-1} \left((1 \blacktriangleright S(Z_{<0>})) (S(Z_{<1>}) \blacktriangleright 1) \right) \\
 &= S_{\blacktriangleleft}^{-1} \left(S(Z_{<1>}) \blacktriangleright 1 \right) S_{\blacktriangleleft}^{-1} \left(1 \blacktriangleright S(Z_{<0>}) \right) \\
 &= \left(Z_{<1>} \blacktriangleright 1 \right) \left(S((S(Z_{<0>}))_{<1>}) \blacktriangleright S((S(Z_{<0>}))_{<0>}) \right) \\
 &= \left(Z_{<1>} \blacktriangleright 1 \right) \left(S((Z_{<0>})_{<1>}) \blacktriangleright Z_{<0> <0>} \right) \\
 &= \left(Z_{<1>(2)} S(Z_{<1>(1)}) \blacktriangleright Z_{<0>} \right) \\
 &= \varepsilon(Z_{<1>}) \blacktriangleright Z_{<0>} \\
 &= 1 \blacktriangleright Z
 \end{aligned}
 \tag{3.25}$$

$$\begin{aligned}
 S_{\blacktriangleleft} S_{\blacktriangleleft}^{-1}(1 \blacktriangleright Z) &= S_{\blacktriangleleft} (S(Z_{<1>}) \blacktriangleright S(Z_{<0>})) \\
 &= \left(1 \blacktriangleright S((S(Z_{<0>}))_{<0>}) \right) \left(S((S(Z_{<0>}))_{<1>}) Z_{<1>} \blacktriangleright 1 \right) \\
 &= \left(1 \blacktriangleright Z_{<0> <0>} \right) \left(S((Z_{<0>})_{<1>}) Z_{<1>} \blacktriangleright 1 \right) \\
 &= \left(1 \blacktriangleright Z_{<0>} \right) \left(S(Z_{<1>(1)}) Z_{<1>(2)} \blacktriangleright 1 \right) \\
 &= \left(1 \blacktriangleright Z_{<0>} \right) \left(\varepsilon(Z_{<1>}) \blacktriangleright 1 \right) \\
 &= 1 \blacktriangleright Z
 \end{aligned}
 \tag{3.26}$$

□

Definition 3.5. *Opposite coalgebra.* For any coalgebra (C, Δ, ε) we set

$$\Delta^{\text{op}} = \tau_{C,C} \circ \Delta$$

where $\tau_{C,C}$ is the twist map. Then $(C, \Delta^{\text{op}}, \varepsilon)$ is a coalgebra which we call the opposite coalgebra and denote by C^{cop} .

Corollary 3.6. *The opposite coalgebra $\mathcal{H} = (\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1))^{\text{cop}}$ is a Hopf algebra with coproduct $\Delta = \Delta_{\blacktriangleleft}^{\text{op}}$ and antipode $S = S_{\blacktriangleleft}^{-1}$.*

Proof. It was proved in [22, Corollary III.3.5] that if the antipode of a Hopf algebra has an inverse, the opposite coalgebra is also a Hopf algebra with opposite coproduct and inverse antipode. □

Definition 3.7. *Let two Lie groups (G_1, G_2) be a matched pair, define \mathcal{H} to be the opposite coalgebra $(\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1))^{\text{cop}}$, with product given by the same of $(\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1))$ (3.18), coproduct given by $\Delta_{\blacktriangleleft}^{\text{op}}$ and antipode $S_{\blacktriangleleft}^{-1}$.*

From now on, we will focus on the opposite coalgebra \mathcal{H} , which we will use to construct a Hopf action later.

3.2. Standard action. Next we would like to construct a Hopf action of \mathcal{H} on \mathcal{A} :

Again take a PBW basis of $\mathcal{U}(\mathfrak{g}_1)$, $\{Z_I = Z_1^{i_1} \cdots Z_p^{i_p}, I = (i_1, \dots, i_p)\}$. Every $Z_I \in \mathcal{U}(\mathfrak{g}_1)$ acts on \mathcal{A} by

$$(3.27) \quad \widetilde{Z}_I(fU_\psi^*) = \widetilde{Z}_1^{i_1} \cdots \widetilde{Z}_p^{i_p}(f)U_\psi^*,$$

$$(3.28) \quad \widetilde{Z}_I(f)U_\psi^*|_{\varphi_0} = \underbrace{\frac{d}{dt_1}\bigg|_{t_1=0}}_{i_1 \text{ times}} \cdots \underbrace{\frac{d}{dt_p}\bigg|_{t_p=0}}_{i_p \text{ times}} f(\varphi_0 \underbrace{\exp(t_1 Z_1) \cdots \exp(t_p Z_p)}_{i_1 \text{ times}} \cdots \underbrace{\exp(t_p Z_p) \cdots}_{i_p \text{ times}})U_\psi^*,$$

on each fU_ψ^* and extend to the finite sums.

$F \in \mathcal{R}(G_2)$ act on \mathcal{A} by

$$(3.29) \quad \widetilde{F}(fU_\psi^*)(\varphi) = F(\psi \triangleleft \varphi)f(\varphi)U_\psi^*,$$

on each fU_ψ^* and extend to the finite sums.

Therefore we would like $F \blacktriangleright Z_I$ to act on \mathcal{A} by

$$(3.30) \quad F \widetilde{\blacktriangleright Z_I}(fU_\psi^*)(\varphi) = F(\psi \triangleleft \varphi) \widetilde{Z}_1^{i_1} \cdots \widetilde{Z}_p^{i_p}(f)(\varphi)U_\psi^*,$$

or equivalently,

$$(3.31) \quad F \widetilde{\blacktriangleright Z_I}(fU_\psi^*)(\varphi) = \varphi \triangleright F(\psi) \widetilde{Z}_1^{i_1} \cdots \widetilde{Z}_p^{i_p}(f)(\varphi)U_\psi^*.$$

Proposition 3.8. *The formula*

$$(3.32) \quad F \widetilde{\blacktriangleright Z_I}(fU_\psi^*)(\varphi) = F(\psi \triangleleft \varphi) \widetilde{Z}_1^{i_1} \cdots \widetilde{Z}_p^{i_p}(f)(\varphi)U_\psi^*$$

defines a Hopf action.

Proof. We need to check that the action is well-defined, i.e., commutes with product and coproduct.

First, we want to show it commutes with product. Because of the formula above, it is trivial that

$$(3.33) \quad \begin{aligned} (\widetilde{F \blacktriangleright 1})(\widetilde{1 \blacktriangleright Z_I}) &= \widetilde{F \blacktriangleright Z_I}, \\ (\widetilde{F \blacktriangleright 1})(\widetilde{G \blacktriangleright 1}) &= \widetilde{(FG) \blacktriangleright 1}, \\ (\widetilde{1 \blacktriangleright Z_I})(\widetilde{1 \blacktriangleright Z_J}) &= \widetilde{1 \blacktriangleright Z_I Z_J}. \end{aligned}$$

Therefore we just need to show that

$$(3.34) \quad \widetilde{1 \blacktriangleright Z_I} \widetilde{F \blacktriangleright 1} = \widetilde{(1 \blacktriangleright Z_I)(F \blacktriangleright 1)}.$$

This is true because

$$\begin{aligned}
& \widetilde{Z_I} \widetilde{F}(fU_\psi^*)(\varphi) \\
&= \widetilde{Z_I}(F(\psi \triangleleft \varphi) f(\varphi) U_\psi^*) \\
&= F(\psi \triangleleft \varphi) (\widetilde{(Z_I)_{(1)} f}(\varphi) U_\psi^* + \frac{d}{dt} \Big|_{t=0} F(\psi \triangleleft (\varphi \exp t(Z_I)_{(2)})) f(\varphi) U_\psi^* \\
(3.35) \quad &= F(\psi \triangleleft \varphi) (\widetilde{(Z_I)_{(1)} f}(\varphi) U_\psi^* + \frac{d}{dt} \Big|_{t=0} F((\psi \triangleleft \varphi)(\exp t(Z_I)_{(2)})) f(\varphi) U_\psi^* \\
&= F(\psi \triangleleft \varphi) (\widetilde{(Z_I)_{(1)} f}(\varphi) U_\psi^* + ((Z_I)_{(2)} \triangleright F)(\psi \triangleleft \varphi) f(\varphi) U_\psi^* \\
&= \widetilde{F}(\widetilde{(Z_I)_{(1)} f}(\varphi) U_\psi^*) + (Z_I)_{(2)} \triangleright F(fU_\psi^*)(\varphi).
\end{aligned}$$

Next, we verify on generators that the action commutes with coproduct,

$$\begin{aligned}
& \widetilde{F}(fU_{\psi_1}^* * gU_{\psi_2}^*)(\varphi) \\
&= \widetilde{F}(f(g \circ \widetilde{\psi_1}) U_{\psi_2 \psi_1}^*)(\varphi) \\
&= F((\psi_2 \psi_1) \triangleleft \varphi) f(\varphi) (g \circ \widetilde{\psi_1}(\varphi)) U_{\psi_2 \psi_1}^* \\
(3.36) \quad &= F(((\psi_2 \triangleleft (\psi_1 \triangleright \varphi)) \psi_1 \triangleleft \varphi) f(\varphi) (g \circ \widetilde{\psi_1}(\varphi)) U_{\psi_2 \psi_1}^* \\
&= F_{(1)}(\psi_2 \triangleleft (\psi_1 \triangleright \varphi)) F_{(2)}(\psi_1 \triangleleft \varphi) f(\varphi) (g \circ \widetilde{\psi_1}(\varphi)) U_{\psi_2 \psi_1}^* \\
&= (F_{(1)}(\psi_2 \triangleleft \varphi) \circ \widetilde{\psi_1}) F_{(2)}(\psi_1 \triangleleft \varphi) f(\varphi) (g \circ \widetilde{\psi_1}(\varphi)) U_{\psi_2 \psi_1}^* \\
&= \widetilde{F}_{(2)}(fU_{\psi_1}^*) * \widetilde{F}_{(1)}(gU_{\psi_2}^*)(\varphi) \\
&= \Delta_{\blacktriangleright \triangleleft}^{\text{op}}(\widetilde{F \blacktriangleright \triangleleft 1})(fU_{\psi_1}^* \otimes gU_{\psi_2}^*)(\varphi),
\end{aligned}$$

$$\begin{aligned}
& \widetilde{Z_i}(fU_{\psi_1}^* * gU_{\psi_2}^*) \\
&= \widetilde{Z_i}(f(g \circ \widetilde{\psi_1}) U_{\psi_2 \psi_1}^*) \\
&= \widetilde{Z_i}(f)(g \circ \widetilde{\psi_1}) U_{\psi_2 \psi_1}^* + f \widetilde{Z_i}(g \circ \widetilde{\psi_1}) U_{\psi_2 \psi_1}^* \\
(3.37) \quad &= \widetilde{Z_i}(f)(g \circ \widetilde{\psi_1}) U_{\psi_2 \psi_1}^* + f \Gamma_i^j(\psi_1) (\widetilde{Z_j}(g) \circ \widetilde{\psi_1}) U_{\psi_2 \psi_1}^* \\
&= \widetilde{Z_i}(fU_{\psi_1}^*) * gU_{\psi_2}^* + (\Gamma_i^j(\psi_1) f) U_{\psi_1}^* * \widetilde{Z_j}(gU_{\psi_2}^*) \\
&= \widetilde{Z_i}(fU_{\psi_1}^*) * gU_{\psi_2}^* + \widetilde{\gamma_i^j}(fU_{\psi_1}^*) * \widetilde{Z_j}(gU_{\psi_2}^*) \\
&= \Delta_{\blacktriangleright \triangleleft}^{\text{op}}(\widetilde{1 \blacktriangleright \triangleleft Z_i})(fU_{\psi_1}^* \otimes gU_{\psi_2}^*)(\varphi).
\end{aligned}$$

□

Note 3.9. The formula also defines a Hopf action on $\mathcal{A}_\Gamma = C_c^\infty(G_1) \rtimes \Gamma$ as well for any discrete subgroup $\Gamma < G_2^\delta$.

In order to do Hopf cyclic theory, we want to show that this Hopf action is equipped a δ -invariant σ -trace. We introduce a functional on the discrete crossed product $\mathcal{A} =$

$C_c^\infty(G_1) \rtimes G_2^\delta$:

$$(3.38) \quad \tau(fU_\psi^*) = \begin{cases} 0, & \text{if } \psi \neq 1, \\ \int_{G_1} f\varpi. & \end{cases}$$

Here ϖ is the volume form attached to the dual basis $\tilde{\omega}_i$ of \tilde{Z}_i on G_1

$$\varpi = \bigwedge \tilde{\omega}_i,$$

and is in fact not always invariant under G_2 action, i.e.,

$$(3.39) \quad \begin{aligned} \tilde{\psi}_*(\tilde{Z}_i) &= U_\psi \tilde{Z}_i U_\psi^* = \Gamma_i^j(\psi^{-1}) \tilde{Z}_j, \\ \tilde{\psi}^*(\tilde{\omega}_i) &= (\Gamma_i^j(\psi^{-1}) \circ \tilde{\psi}) \tilde{\omega}_j, \\ \tilde{\psi}^*(\varpi) &= \bigwedge (\Gamma_i^j(\psi^{-1}) \circ \tilde{\psi}) \tilde{\omega}_j = (\det(\Gamma(\psi^{-1})) \circ \tilde{\psi}) \varpi. \end{aligned}$$

Evaluating at identity gives

$$(3.40) \quad \tilde{\psi}^*(\bigwedge \omega_i) = \det(\gamma)(\psi^{-1}) \bigwedge \omega_i.$$

We see a representative function $\sigma := \det(\gamma)$ here. It is easy to see that σ is a group-like element in $\mathcal{R}(G_2)$, i.e., $\sigma(\psi_1\psi_2) = \sigma(\psi_1)\sigma(\psi_2)$.

we also have the module function on \mathfrak{g}_1 , which is the trace of the adjoint representation \mathfrak{g}_1 on itself,

$$\delta(Z) = \text{Tr}(ad_Z),$$

and we can extend it to a character on $\mathcal{U}(\mathfrak{g}_1)$ and then \mathcal{H} by

$$\delta(F \blacktriangleright Z_I) = \varepsilon(F) \blacktriangleright \delta(Z_I),$$

so we have a group-like element $\sigma \blacktriangleright 1$ in \mathcal{H} and a character δ on \mathcal{H} .

Let us recall the definition of modular pairs in involution as follows:

Definition 3.10. Let \mathcal{H} be a Hopf algebra, $\delta : \mathcal{H} \rightarrow \mathbb{C}$ be a character and $\sigma \in \mathcal{H}$ be a group-like element. The pair (δ, σ) is called a modular pair in involution (MPI) if

$$(3.41) \quad \delta(\sigma) = 1, \quad S_\delta^2(h) = \sigma h \sigma^{-1},$$

for all $h \in \mathcal{H}$, where S_δ is the convolution $\delta * S$:

$$S_\delta(h) = \delta(h_{(1)})S(h_{(2)}).$$

A modular pair in involution is the easiest coefficient that we can put for Hopf cyclic cohomology. The generalized coefficients are called stable anti-Yetter-Drinfeld (SAYD) module over \mathcal{H} .

Assume \mathcal{H} is a Hopf algebra with invertible antipode, we can define Yetter-Drinfeld module over \mathcal{H} as in [4, p. 181]. It is shown in [17] that if we replace S with S^{-1} , S^{-1} with S in the definition of Yetter-Drinfeld module we can have compatible action and coaction. The new definition gives the so called anti-Yetter-Drinfeld module over \mathcal{H} .

Definition 3.11. Let \mathcal{H} be a Hopf algebra with invertible antipode. A right \mathcal{H} -module, left \mathcal{H} -comodule M is called a right-left anti-Yetter-Drinfeld module over \mathcal{H} if

$$(3.42) \quad \blacktriangledown(m \triangleleft h) = S(h_{(3)})m_{<-1>}h_{(1)} \otimes m_{<0>} \triangleleft h_{(2)}, \quad \forall m \in M, h \in \mathcal{H},$$

moreover, M is called stable if

$$(3.43) \quad m_{<0>} \triangleleft m_{<-1>} = m.$$

Lemma 3.12. [17, Lemma 2.2] *Let the ground field \mathbb{C} be a right module over \mathcal{H} via a character δ and a left comodule over H via a group-like σ . Then ${}^\sigma\mathbb{C}_\delta$ is a stable right-left anti-Yetter-Drinfeld module if and only if (δ, σ) is a modular pair in involution.*

Proof. It's a Lemma in [17]. However we would like to give an illustration here: The second (stability) condition is obvious, so we will just show the first one. On one side, if

$$S_\delta^2(h) = \sigma h \sigma^{-1},$$

we calculate

$$(3.44) \quad \begin{aligned} S(h_{(3)})1_{<-1>}h_{(1)} \otimes 1_{<0>} \cdot h_{(2)} &= S(h_{(3)})\sigma h_{(1)} \otimes \delta(h_{(2)}) \\ &= S_\delta(h_{(2)})\sigma h_{(1)}\sigma^{-1}\sigma \otimes 1 \\ &= S_\delta(h_{(2)})S_\delta^2(h_{(1)})\sigma \otimes 1 \\ &= S_\delta(S_\delta(h_{(1)})h_{(2)})\sigma \otimes 1 \\ &= S_\delta(\delta(h_{(1)})S(h_{(2)})h_{(3)})\sigma \otimes 1 \\ &= S_\delta(\delta(h)\eta)\sigma \otimes 1 \\ &= \sigma \otimes \delta(h) \\ &= \blacktriangledown(1 \cdot h). \end{aligned}$$

On the other side, if

$$\sigma \otimes \delta(h) = S(h_{(3)})\sigma h_{(1)} \otimes \delta(h_{(2)}),$$

then

$$\delta(h) = \sigma^{-1}S_\delta(h_{(2)})\sigma h_{(1)},$$

we calculate

$$(3.45) \quad \begin{aligned} h &= \delta(h_{(1)})S(h_{(2)})h_{(3)} \\ &= \delta(h_{(1)})\sigma^{-1}S_\delta((S(h_{(2)}))_{(2)})\sigma(S(h_{(2)}))_{(1)}h_{(3)} \\ &= \delta(h_{(1)})\sigma^{-1}S_\delta(S(h_{(2)(1)}))\sigma(S(h_{(2)(2)}))h_{(3)} \\ &= \sigma^{-1}S_\delta(\delta(h_{(1)})S(h_{(2)}))\sigma\varepsilon(h_{(3)})\eta \\ &= \sigma^{-1}S_\delta^2(h)\sigma. \end{aligned}$$

□

We also have

Lemma 3.13. *If ${}^\sigma\mathbb{C}_\delta$ is a right-left stable anti-Yetter-Drinfeld module over \mathcal{H} , then ${}^{S^{-1}(\sigma)}\mathbb{C}_\delta$ is a right-left stable anti-Yetter-Drinfeld module over \mathcal{H}^{cop} .*

Proof. straightforward calculation. Apply S^{-1} to equation 3.42. □

Now we are ready to show that the pair we constructed is a modular pair in involution.

Proposition 3.14. *The pair (δ, σ) is a modular pair in involution for the Hopf algebra \mathcal{H} .*

Proof. [34, Thm 3.2] shows that $(\delta, S(\det(\gamma)) = \sigma^{-1})$ is a modular pair in involution for the Hopf algebra $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$ (we have used antipode in our right coaction so their σ is our σ^{-1}). By 3.12, $\sigma^{-1}\mathbb{C}_\delta$ is a SAYD over $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)$. Use previous lemma, ${}^\sigma\mathbb{C}_\delta$ is a right-left SAYD over $\mathcal{R}(G_2) \blacktriangleright \mathcal{U}(\mathfrak{g}_1)^{\text{cop}}$ and (δ, σ) is a modular pair in involution for the Hopf algebra \mathcal{H} . \square

Let us now calculate S_δ on these generators for future use.

$$\begin{aligned}
(3.46) \quad S_\delta(1 \blacktriangleright Z_i) &= \mu(\delta \otimes S)\Delta(1 \blacktriangleright Z_i) \\
&= \mu(\delta \otimes S_{\blacktriangleright}^{-1})\Delta_{\blacktriangleright}^{\text{op}}(1 \blacktriangleright Z_i) \\
&= \mu(\delta \otimes S_{\blacktriangleright}^{-1})(1 \blacktriangleright Z_i \otimes 1 \blacktriangleright 1 + S(\gamma_i^j) \blacktriangleright 1 \otimes 1 \blacktriangleright Z_j) \\
&= \delta(Z_i)1 \blacktriangleright 1 + \delta_i^j S_{\blacktriangleright}^{-1}(1 \blacktriangleright Z_j) \\
&= \delta(Z_i)1 \blacktriangleright 1 + S_{\blacktriangleright}^{-1}(1 \blacktriangleright Z_i) \\
&= \delta(Z_i)1 \blacktriangleright 1 + \gamma_i^j \blacktriangleright S(Z_j),
\end{aligned}$$

and

$$\begin{aligned}
(3.47) \quad S_\delta(F \blacktriangleright 1) &= \mu(\delta \otimes S)\Delta(F \blacktriangleright 1) \\
&= \mu(\delta \otimes S_{\blacktriangleright}^{-1})\Delta_{\blacktriangleright}^{\text{op}}(F \blacktriangleright 1) \\
&= \mu(\delta \otimes S_{\blacktriangleright}^{-1})(F_{(2)} \blacktriangleright 1) \otimes (F_{(1)} \blacktriangleright 1) \\
&= \varepsilon(F_{(2)})S(F_{(1)}) \blacktriangleright 1 \\
&= S(F) \blacktriangleright 1.
\end{aligned}$$

With a module pair in involution in hand we can associate a cyclic structure to \mathcal{H} as suggested in [9]. Specifically, we set $C^n(\mathcal{H}) = \mathcal{H}^{\otimes n}$ for $n \geq 1$, and $C^0(\mathcal{H}) = \mathbb{C}$. The face maps $\delta_i : C^{n-1}(\mathcal{H}) \rightarrow C^n(\mathcal{H})$, $0 \leq i \leq n$, are:

$$\begin{aligned}
(3.48) \quad \delta_0(h^1 \otimes \cdots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \cdots \otimes h^{n-1}, \\
\delta_i(h^1 \otimes \cdots \otimes h^{n-1}) &= h^1 \otimes \cdots \otimes \Delta(h^i) \otimes \cdots \otimes h^{n-1}, \quad 1 \leq i \leq n-1, \\
\delta_n(h^1 \otimes \cdots \otimes h^{n-1}) &= h^1 \otimes \cdots \otimes h^{n-1} \otimes \sigma,
\end{aligned}$$

for $n > 1$, and

$$\delta_0(1) = 1, \quad \delta_1(1) = \sigma,$$

for $n = 1$.

The degeneracy maps $\sigma_i : C^{n+1}(\mathcal{H}) \rightarrow C^n(\mathcal{H})$, $0 \leq i \leq n$, are:

$$(3.49) \quad \sigma_i(h^1 \otimes \cdots \otimes h^{n+1}) = h^1 \otimes \cdots \otimes \varepsilon(h^{i+1}) \otimes \cdots \otimes h^{n+1},$$

for $n > 0$, and

$$\sigma_0(h) = \varepsilon(h),$$

for $n = 0$.

The cyclic operator $\tau_n : C^n(\mathcal{H}) \rightarrow C^n(\mathcal{H})$, is defined as:

$$\begin{aligned}
(3.50) \quad \tau_n(h^1 \otimes \cdots \otimes h^n) &= (\Delta^{n-1} S_\delta(h^1)) \cdot h^2 \otimes \cdots \otimes h^n \otimes \sigma, \\
&= \sum S(h_{(n)}^1) h^2 \otimes \cdots \otimes S(h_{(2)}^1) h^n \otimes S_\delta(h_{(1)}^1) \sigma.
\end{aligned}$$

The periodic Hopf cyclic cohomology $HP^\bullet(\mathcal{H};^\sigma \mathbb{C}_\delta)$ of \mathcal{H} with coefficients in the modular pair (δ, σ) is, by definition (cf. [7, 8]), the \mathbb{Z}_2 -graded cohomology of the total complex $CC^\bullet(\mathcal{H};^\sigma \mathbb{C}_\delta)$ associated to the bicomplex $\{CC^\bullet(\mathcal{H};^\sigma \mathbb{C}_\delta), b, B\}$, where

$$\begin{aligned}
 CC^{p,q}(\mathcal{H};^\sigma \mathbb{C}_\delta) &= \begin{cases} 0, & \text{if } p < q, \\ C^{p-q}(\mathcal{H};^\sigma \mathbb{C}_\delta), & \text{if } q \geq p, \end{cases} \\
 b : C^m(\mathcal{H};^\sigma \mathbb{C}_\delta) &\rightarrow C^{m+1}(\mathcal{H};^\sigma \mathbb{C}_\delta) \\
 b &= \sum_{i=0}^n (-1)^i \delta_i, \\
 B : C^m(\mathcal{H};^\sigma \mathbb{C}_\delta) &\rightarrow C^{m-1}(\mathcal{H};^\sigma \mathbb{C}_\delta) \\
 B &= \left(\sum_{i=0}^{n-1} (-1)^{(n-1)i} \tau_{n-1}^i \right) \sigma_{n-1} \tau_n (1 - (-1)^n \tau_n).
 \end{aligned}
 \tag{3.51}$$

Proposition 3.15 ([7, 8]). *Let \mathcal{H} be a Hopf algebra endowed with a modular pair in involution (δ, σ) . Then $\mathcal{H}_{(\delta, \sigma)}^\natural = \{C^n(\mathcal{H})\}_{n \geq 0}$ equipped with the operators given by 3.48 to 3.50 is a module over the cyclic category Λ .*

Proof. See [9, Thm 1] for proof. Note it is the MPI condition 3.41 that ensured the cyclicity. \square

Now we will take a deeper look at the Hopf action and the trace we just defined.

Proposition 3.16. *For any $a, b \in \mathcal{A}$ and $h \in \mathcal{H}$ one has*

$$\begin{aligned}
 \tau(ab) &= \tau(b\sigma(a)), \\
 \tau(h(a)) &= \delta(h)\tau(a), \\
 \tau(h(a)b) &= \tau(aS_\delta(h)(b)),
 \end{aligned}
 \tag{3.52}$$

hence τ is a δ -invariant σ -trace under the Hopf action.

Proof. It suffices to verify the identities for non trivial cases. i.e., we assume $a = fU_\psi^*$ and $b = gU_\psi$ (recall from our convention 3.1, $U_\psi = U_{\psi^{-1}}^*$) for the first and third identities and $a = fU_1^*$ for the second one.

$$\begin{aligned}
 \tau(fU_\psi^* * gU_\psi) &= \int_{G_1} f(g \circ \widetilde{\psi}) \varpi \\
 &= \int_{G_1} (f \circ \widetilde{\psi^{-1}}) g \widetilde{\psi^{-1}}^*(\varpi) \\
 &= \int_{G_1} (f \circ \widetilde{\psi^{-1}}) g (\det(\Gamma)(\psi) \circ \widetilde{\psi^{-1}}) \varpi \\
 &= \int_{G_1} (f \circ \widetilde{\psi^{-1}})(\varphi) g(\varphi) \det(\Gamma)(\psi) (\psi^{-1} \triangleright \varphi) \varpi(\varphi) \\
 &= \int_{G_1} (f \circ \widetilde{\psi^{-1}})(\varphi) g(\varphi) \det(\gamma)(\psi \triangleleft (\psi^{-1} \triangleright \varphi)) \varpi(\varphi) \\
 &= \tau(gU_\psi * \widetilde{\det(\gamma)}(fU_\psi^*)).
 \end{aligned}
 \tag{3.53}$$

For the second identity, because of obvious multiplicativity, we verify on generators $1 \blacktriangleright Z_i$:

$$\begin{aligned}
 \int_{G_1} \widetilde{Z}_i(f) \varpi &= \int_{G_1} \left. \frac{d}{dt} \right|_{t=0} f(\varphi \exp(tZ_i)) \varpi(\varphi) \\
 &= \left. \frac{d}{dt} \right|_{t=0} \int_{G_1} f(\varphi \exp(tZ_i)) \varpi(\varphi) \\
 (3.54) \quad &= \left. \frac{d}{dt} \right|_{t=0} \int_{G_1} f(\varphi) \varpi(\varphi \exp(-tZ_i)) \\
 &= \left. \frac{d}{dt} \right|_{t=0} \Delta(tZ_i) \int_{G_1} f(\varphi) \varpi(\varphi) \\
 &= \delta(Z_i) \int_{G_1} f \varpi,
 \end{aligned}$$

and on $F \blacktriangleright 1$:

$$(3.55) \quad \int_{G_1} F((1 \triangleleft \varphi)) f(\varphi) \varpi(\varphi) = F(1) \int_{G_1} f \varpi.$$

For the third identity, because of anti-multiplicativity of S_δ , we verify on generators $F \blacktriangleright 1$:

$$\begin{aligned}
 \tau(\widetilde{F \blacktriangleright 1}(fU_\psi^*) * gU_\psi) &= \int_{G_1} F((\psi \triangleleft \varphi)) f(\varphi) g(\psi \triangleright \varphi) \varpi(\varphi) \\
 &= \int_{G_1} F((\psi^{-1} \triangleleft (\psi \triangleright \varphi))^{-1}) f(\varphi) g(\psi \triangleright \varphi) \varpi(\varphi) \\
 (3.56) \quad &= \int_{G_1} f(\varphi) g(\psi \triangleright \varphi) F((\psi^{-1} \triangleleft (\psi \triangleright \varphi))^{-1}) \varpi(\varphi) \\
 &= \int_{G_1} f(\varphi) g(\psi \triangleright \varphi) S(F)((\psi^{-1} \triangleleft (\psi \triangleright \varphi))) \varpi(\varphi) \\
 &= \tau(fU_\psi^* * S_\delta(\widetilde{F \blacktriangleright 1})(gU_\psi)).
 \end{aligned}$$

and on $1 \blacktriangleright Z_i$:

$$\begin{aligned}
(3.57) \quad \tau(\widetilde{1 \blacktriangleright Z_i}(fU_\psi^* * gU_\psi)) &= \int_{G_1} \widetilde{Z_i}(f)(g \circ \tilde{\psi}) \varpi \\
&= \int_{G_1} \frac{d}{dt} \Big|_{t=0} f(\varphi \exp(tZ_i))g(\psi \triangleright \varphi) \varpi(\varphi) \\
&= \frac{d}{dt} \Big|_{t=0} \int_{G_1} f(\varphi)g(\psi \triangleright (\varphi \exp(-tZ_i))) \varpi(\varphi \exp(-tZ_i)) \\
&= \frac{d}{dt} \Big|_{t=0} \int_{G_1} f(\varphi)g(\psi \triangleright (\varphi \exp(-tZ_i))) \varpi(\varphi) \\
&\quad + \frac{d}{dt} \Big|_{t=0} \int_{G_1} f(\varphi)g(\psi \triangleright \varphi) \varpi(\varphi \exp(-tZ_i)) \\
&= \int_{G_1} f(\varphi) (\Gamma_i^j(\psi^{-1}) \circ \tilde{\psi})(\varphi) (-\tilde{Z}_j(g) \circ \tilde{\psi})(\varphi) \varpi(\varphi) \\
&\quad + \frac{d}{dt} \Big|_{t=0} \Delta(tZ_i) \int_{G_1} f(\varphi)g(\psi \triangleright \varphi) \varpi(\varphi) \\
&= \int_{G_1} f(\varphi) \gamma_i^j(\psi^{-1} \triangleleft (\psi \triangleright \varphi)) (-\tilde{Z}_j(g) \circ \tilde{\psi})(\varphi) \varpi(\varphi) \\
&\quad + \delta(Z_i) \int_{G_1} f(\varphi)g(\psi \triangleright \varphi) \varpi(\varphi) \\
&= \tau(fU_\psi^* * ((1 \blacktriangleright \gamma_i^j)(\widetilde{S(Z_j)} \blacktriangleright 1)(gU_\psi))) \\
&\quad + \delta(Z_i) \tau(fU_\psi^* * gU_\psi) \\
&= \tau(fU_\psi^* * ((1 \blacktriangleright \gamma_i^j)(\widetilde{S(Z_j)} \blacktriangleright 1)(gU_\psi))) \\
&\quad + \tau(fU_\psi^* * \widetilde{\delta(Z_i)}(gU_\psi)) \\
&= \tau(fU_\psi^* * \widetilde{S_\delta(Z_i \blacktriangleright 1)}(gU_\psi)),
\end{aligned}$$

□

Proposition 3.17 ([7, 8]). *Let $\tau : \mathcal{A} \rightarrow \mathbb{C}$ be a δ -invariant σ -trace under the Hopf action of \mathcal{H} on \mathcal{A} . Then the assignment*

$$(3.58) \quad \lambda(h^1 \otimes \cdots \otimes h^n)(a^0, \dots, a^n) = \tau(a^0 h^1(a^1) \cdots h^n(a^n))$$

defines a map of Λ -modules $\lambda^\natural : \mathcal{H}_{(\delta, \sigma)}^\natural \rightarrow \mathcal{A}^\natural$, which induces characteristic homomorphisms in cyclic cohomology:

$$\lambda_\tau^* : HC_{(\delta, \sigma)}^*(\mathcal{H}) \rightarrow HC^*(\mathcal{A}).$$

Corollary 3.18. *For any discrete subgroup $\Gamma < G_2^\delta$, $\tau : \mathcal{A}_\Gamma \rightarrow \mathbb{C}$ is also a δ -invariant σ -trace under the Hopf action of \mathcal{H} on \mathcal{A}_Γ . The assignment above also defines a map of Λ -modules $\lambda^\natural : \mathcal{H}_{(\delta, \sigma)}^\natural \rightarrow \mathcal{A}_\Gamma^\natural$, which induces characteristic homomorphisms in cyclic cohomology:*

$$\lambda_\tau^* : HC_{(\delta, \sigma)}^*(\mathcal{H}) \rightarrow HC^*(\mathcal{A}) \rightarrow HC^*(\mathcal{A}_\Gamma).$$

Theorem 3.19. *Let (G_1, G_2) be a matched pair of Lie groups, \mathcal{A} be the convolution algebra $C_c^\infty(G_1) \rtimes G_2^\delta$. $\tau : \mathcal{A} \rightarrow \mathbb{C}$ be a δ -invariant σ -trace, as defined in this section, under the Hopf action of $\mathcal{H} = (\mathcal{U}(\mathfrak{g}_1) \bowtie \mathcal{R}(G_2))^{\text{cop}}$ on \mathcal{A} . Then the assignment*

$$(3.59) \quad \lambda(h^1 \otimes \cdots \otimes h^n)(a^0, \dots, a^n) = \tau(a^0 h^1(a^1) \cdots h^n(a^n)), \quad h^i \in \mathcal{H}, \quad a^i \in \mathcal{A},$$

defines an injective homomorphism of Λ -modules $\lambda^\sharp : \mathcal{H}_{(\delta, \sigma)}^\sharp \rightarrow \mathcal{A}^\sharp$.

Before we go to the proof we list two useful lemmas.

Lemma 3.20. *the σ -trace τ is faithful, i.e., that*

$$\tau(ab) = 0, \quad \forall a \in \mathcal{A}, \quad \text{implies } b = 0.$$

Proof. We just need to show that if $\int_{G_1} f(g \circ \tilde{\psi}) \varpi = 0$ for all g , then $f = 0$, which is the same that if $\int_{G_1} fg \varpi = 0$ for all g , then $f = 0$.

Now if we assume $\int_{G_1} fg \varpi = 0$ for all g , but $f(p) \neq 0$ at some $p \in G_1$. Then there exists some small neighborhood U of p such that f is strictly positive or negative, say $f > c > 0$ on U . We can have a compact set $V_1 \subset U$ and a smaller compact set $V_2 \subsetneq V_1$. Take g to be a smooth cutoff function that is identity on V_2 and vanish off V_1 , then $\int_{G_1} fg \varpi = \int_{V_1} fg \varpi > \int_{V_2} fg \varpi > \int_{V_2} c \varpi > 0$, hence a contradiction. \square

Lemma 3.21.

$$h(a)(e) = 0, \quad \forall a \in \mathcal{A}, \quad \text{implies } h = 0.$$

Proof. Recall 3.2 and 3.21, since the Z_I 's form a PBW basis of $\mathcal{U}(\mathfrak{g}_1)$, and that every element $\sum_j F_j \bowtie u_j$ in $\mathcal{R}(G_2) \bowtie \mathcal{U}(\mathfrak{g}_1)$ can be written uniquely as

$$(3.60) \quad \sum_j F_j \bowtie u_j = \sum_I F_I \bowtie Z_I, \quad I = (i_1, \dots, i_p),$$

we just need to prove that if

$$\sum \widetilde{F_I} \widetilde{Z_I}(fU_\psi^*)(e) = 0, \quad \forall fU_\psi^* \in \mathcal{A},$$

then $F_I = 0$ for any I .

However, for every $\psi \in G_2$ we will have

$$\sum F_I(\psi \triangleleft e) \widetilde{Z_I}(f)(e) = 0, \quad \forall f \in C_c^\infty(G_1),$$

Because Z_I form a PBW basis of $\mathcal{U}(\mathfrak{g}_1)$, we order the sum by the order of the PBW basis assume that the coefficient $F_{I_n}(\psi)$ is non-zero for I_n with top degree. There exists a function $g_n \in C_c^\infty(G_1)$ defined on a neighborhood of e such that $\widetilde{Z_{I_n}}(g_n)(e) = 1$ and $\widetilde{Z_{I_s}}(g_n)(e) = 0$ for all lower $s < n$. Therefore we have $F_{I_n}(\psi) = 0$, a contradiction. Therefore $F_I(\psi) = 0$ for any I . Since this is true for every $\psi \in G_2$, we then have our desired conclusion. \square

Proof. Now we are ready to prove theorem 3.19.

Assume $\tau(\sum_j a^0 h_j^1(a^1) \cdots h_j^n(a^n)) = 0$ for any a^0, \dots, a^n , by lemma 3.20, we conclude

$$(3.61) \quad \sum_j h_j^1(a^1) \cdots h_j^n(a^n) = 0,$$

when evaluating at the identity, the above equation becomes

$$(3.62) \quad \sum_j h_j^1(a^1) \cdots h_j^n(a^n)(e) = 0,$$

from where we want to show

$$\sum_j h_j^1 \otimes \cdots \otimes h_j^n = 0.$$

We use induction: When $n = 1$, lemma 3.21 applies. Now assume the statement is true of $n - 1$, i.e., if

$$\sum_j h_j^1(a^1) \cdots h_j^{n-1}(a^{n-1})(e) = 0,$$

we have

$$(3.63) \quad \sum_j h_j^1 \otimes \cdots \otimes h_j^{n-1} = 0.$$

Denote $a^k = f_k U_{\psi_k}^*$. If we use 3.21 and the PBW basis of $\mathcal{U}(\mathfrak{g}_1)$ and write each h_j^k as $F_{j,I(j,k)}^k \blacktriangleright Z_{j,I(j,k)}^k$. Since e is fixed by every $\psi \in G_2$, when evaluating at the identity the above equality becomes

$$\sum_{j,I} F_{j,I(j,1)}^1((\psi_1 \triangleleft e)) \widetilde{Z_{j,I(j,1)}^1}(f_1)(e) \cdots F_{j,I(j,n)}^n((\psi_n \triangleleft e)) \widetilde{Z_{j,I(j,n)}^n}(f_n)(e) = 0.$$

We partition the sum by the order of $Z_{j,I(j,n)}^n$ and index with $(1 \leq s \leq N, Q(s))$ and use induction again on s . after reordering the above equality becomes

$$\sum_{I_1 \leq I_s \leq I_N} F_{I_s}^n(\psi_n) \widetilde{Z_{I_s}^n}(f_n)(e) \left(\sum_{Q(s)} F_Q^1(\psi_1) \widetilde{Z_Q^1}(f_1)(e) \cdots F_Q^{n-1}(\psi_{n-1}) \widetilde{Z_Q^{n-1}}(f_{n-1})(e) \right) = 0.$$

Now if we look at the top $Z_{I_N}^n$ with nonzero F_{I_N} , because of the PBW basis, there exists a function $g_n \in C_c^\infty(G_1)$ defined on a neighborhood of e such that $\widetilde{Z_{I_N}^n}(g_n)(e) = 1$ and $\widetilde{Z_{I_s}^n}(g_n)(e) = 0$ for all lower $s < N$. Since we assume F_{I_N} is nonzero, there also exist some ψ_n^+ such that $F_{I_N}(\psi_n^+) \neq 0$.

Now if we fix $a^n = g_n U_{\psi_n^+}^*$, the above equality becomes

$$\sum_{Q(N)} F_Q^1(\psi_1) \widetilde{Z_Q^1}(f_1)(e) \cdots F_Q^{n-1}(\psi_{n-1}) \widetilde{Z_Q^{n-1}}(f_{n-1})(e) = 0 \quad \forall f_k U_{\psi_k}^*, \quad 1 \leq k \leq n-1,$$

this is the induction hypothesis for $n - 1$ of the first induction, hence we have

$$(3.64) \quad \sum_{Q(N)} h_Q^1 \otimes \cdots \otimes h_Q^{n-1} = 0.$$

Therefore

$$\left(\sum_{Q(N)} h_Q^1 \otimes \cdots \otimes h_Q^{n-1} \right) \otimes h_N^n = 0.$$

This means the top degree part, with the new order $(1 \leq s \leq N, Q(s))$, of the original sum

$$\sum_j h_j^1 \otimes \cdots \otimes h_j^n$$

is 0. By the reversed induction on the degree, the whole term is 0 and λ is injective. \square

Definition 3.22. An n -cochain φ on the algebra $C_c^\infty(G_1) \rtimes G_2^\delta$ is representative iff it is in the range of the above monomorphism λ . We denote the representative cochain space as $\text{Im}(\lambda^\natural)$.

Corollary 3.23. Under the assumption of the above theorem, λ is an isomorphism of Λ -modules $\lambda^\natural : \mathcal{H}_{(\delta, \sigma)}^\natural \rightarrow \text{Im}(\lambda^\natural)$.

From now on we do not distinct between $\mathcal{H}_{(\delta, \sigma)}^\natural$ and $\text{Im}(\lambda^\natural)$.

4. DIFFERENTIABLE AND REPRESENTATIVE COHOMOLOGIES OF ACTION GROUPOIDS

In this section, we will first review Haefliger's differentiable cohomology of a action groupoid [16], in particular we take our $G_1 \rtimes G_2^\delta$ as example. Then we refine the complex that calculate the differentiable cohomology and define representative cohomology of a action groupoid. The differentiable and representative cohomology works in general for any action groupoid. However, for our example $G_1 \rtimes G_2^\delta$ with a matched pair of Lie groups (G_1, G_2) , There is an even smaller subcomplex which we called $D^{p,q}$ that satisfies some strong covariance property. We will explain in detail the description of $D^{p,q}$ and its relation (as appeared in the last column of diagram 1.2) between the representative cochain space $\text{Im}(\lambda^\natural)$ from previous section via Conne's Φ map.

4.1. Decomposition of Lie groups. Recall from [20] that a *nucleus* of a Lie group G is a simply connected solvable closed normal subgroup L of G such that G/L is reductive, in the sense that G/L has a faithful representation and every finite dimensional analytic representation of G/L is semisimple. In this case one proves ([20, Thm. 9.1]) that $G = H \rtimes L$, where $H \cong G/L$ is the complementary closed subgroup which is reductive. We will assume all groups are connected through this paper. H acts on L from the right by

$$(4.1) \quad l \triangleleft h = h^{-1}lh$$

which coincide with the right multiplication of H on $H \backslash G$

$$(4.2) \quad (Hl)h = H(hh^{-1}lh) = H(l \triangleleft h)$$

Let, in addition, $\mathfrak{h}, \mathfrak{l} \subset \mathfrak{g}$ be Lie algebras of H, L and G respectively.

Parallel decomposition of Linear algebraic group is called the Levi decomposition ([19]) where $G = H \rtimes L$, H the levi factor and $L = G_u$ the unipotent radical.

For a matched pair of Lie groups G_1 and G_2 we can form the bicrossed product group $G_1 \bowtie G_2$ with group structure given by

$$(\varphi_1 \bowtie \psi_1)(\varphi_2 \bowtie \psi_2) = (\varphi_1 \psi_1 \triangleright \varphi_2) \bowtie (\psi_1 \triangleleft \varphi_2 \psi_2).$$

For a matched pair of Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 we can also form the bicrossed product Lie algebra $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ with underlying vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and bracket relation:

$$[X_1 \oplus Y_1, X_2 \oplus Y_2] = ([X_1, x_2] + Y_1 \triangleright X_2 - Y_2 \triangleright X_1) \oplus ([Y_1, Y_2] + Y_1 \triangleleft X_2 - Y_2 \triangleleft X_1).$$

Now let $G_2 = H_2 \ltimes L_2$ be such decomposition of G_2 . In general, the simply-connected nucleus L_2 of a Lie group is differ from nilpotent by a vector group, to simplify the situation we assume further L_2 is nilpotent with Lie algebra \mathfrak{l}_2 . In the algebraic case L_2 as the unipotent radical is nilpotent. Under our assumption, for both case we have $L_2 = \exp(\mathfrak{l}_2)$ and $\mathfrak{l}_2 = \mathfrak{nil}(\mathfrak{g}_2)$.

4.2. Differentiable cohomology of action groupoid. Let $\mathcal{G} = G_1 \ltimes G_2^\delta$ be our action groupoid, $\Omega_{G_1}^*$ be the complex of differential forms on G_1 . We introduce the following double complex $C^{p,q}, d_1, d_2$. $C^{p,q} = \{0\}$ unless $p \geq 0$ and $0 \leq q \leq \dim G_1$, $C^{p,q}$ be the space of totally antisymmetric maps $\alpha : G_2^{p+1} \rightarrow \Omega^q(G_1)$ that satisfy the G_2 -equivariant condition

$$(4.3) \quad \alpha(\psi_0\psi, \dots, \psi_p\psi) = (\psi^{-1}\triangleright)^* \alpha(\psi_0, \dots, \psi_p), \quad \forall \psi, \psi_i \in G_2.$$

The coboundary $d_1 : C^{p,q} \rightarrow C^{p+1,q}$ is given by

$$(4.4) \quad (d_1\alpha)(\psi_0, \dots, \psi_{p+1}) = \sum_{j=0}^{p+1} (-1)^{q+j} \alpha(\psi_0, \dots, \check{\psi}_j, \dots, \psi_{p+1}),$$

The coboundary $d_2 : C^{p,q} \rightarrow C^{n,q+1}$ is given by de Rham boundary:

$$(4.5) \quad (d_2\alpha)(\psi_0, \dots, \psi_{p+1}) = d\alpha(\psi_0, \dots, \psi_{p+1}).$$

The complex $C^{p,q}$ has a subcomplex $C_d^{p,q}$, which consists of forms $\alpha(\psi_0, \dots, \psi_p)$ that depend on ψ_i *smoothly*, i.e., we can write $\alpha \in C_d^{p,q}$ as

$$(4.6) \quad \alpha(\psi_0, \dots, \psi_p) = \sum c_j \rho_j,$$

here ρ_j form a basis of left G_1 invariant forms on G_1 , and c_j are smooth functions on G_2 .

Definition 4.1. [16] Let $\mathcal{G} = G_1 \ltimes G_2^\delta$. The cohomology of the action groupoid $H^\bullet(\mathcal{G}; \mathbb{R})$ is the cohomology of the simple complex associated to the above double complex $C^{p,q}$ and the differentiable cohomology $H_d^\bullet(\mathcal{G}; \mathbb{R})$ is the cohomology of the simple complex associated to the subcomplex $C_d^{p,q} = C_d^p(G_2^\delta; \Omega_{G_1}^q)$ whose elements are forms depending smoothly on p elements of G_2 as in 4.2.

Proposition 4.2. [16, Thm4.2] Let $\mathcal{G} = G_1 \ltimes G_2^\delta$. $H_d^\bullet(\mathcal{G}; \mathbb{R})$ is isomorphic to the cohomology of G_2 -invariant forms on $G_2/K_2 \times G_1$, where K_2 is a maximal compact subgroup of G_2 , and G_2 acts diagonally.

4.3. Representative cohomology of action groupoid. We refine Haefliger's differentiable cohomology of groupoid by defining a subcomplex, $C_{\mathcal{R}}^p(G_2^\delta; \Omega_{G_1}^q)$ of $C_d^p(G_2^\delta; \Omega_{G_1}^q)$. The elements of $C_{\mathcal{R}}^{p,q} = C_{\mathcal{R}}^p(G_2^\delta; \Omega_{G_1}^q)$ are forms depending *representatively* on $p+1$ elements of G_2 . i.e., the coefficient functions are products of $p+1$ representative functions on G_2 , we can write $\alpha \in C_{\mathcal{R}}^{p,q}$ as

$$(4.7) \quad \alpha(\psi_0, \dots, \psi_p) = \sum c_j \rho_j,$$

where c_j are representative functions on G_2 .

Since representative functions are smooth, we can get a subcomplex and define the *representative cohomology* $H_{\mathcal{R}}^\bullet(\mathcal{G}; \mathbb{R})$:

Definition 4.3. Let $\mathcal{G} = G_1 \rtimes G_2^\delta$. The representative cohomology $H_{\mathcal{R}}^\bullet(\mathcal{G}; \mathbb{R})$ is the cohomology of the simple complex associated to the double complex $C_{\mathcal{R}}^p(G_2^\delta; \Omega_{G_1}^q)$.

Theorem 4.4. Let $\mathcal{G} = G_1 \rtimes G_2^\delta$. The representative cohomology $H_{\mathcal{R}}^\bullet(\mathcal{G}; \mathbb{R})$ is isomorphic to the cohomology of G_2 -invariant forms on $L_2 \times G_1$, where L_2 is the nucleus of G_2 , and G_2 acts diagonally.

Proof. Step 1:

Consider the first inclusion

$$\varepsilon_1 : C_{\mathcal{R}}^p(G_2^\delta; \Omega_{G_1}^q) \rightarrow C_{\mathcal{R}}^p(G_2^\delta; \Omega_{(L_2 \times G_1)}^q)$$

induced by the projection $\pi : L_2 \times G_1 \rightarrow G_1$ induces an isomorphism of the cohomology of the double complexes.

By a familiar spectral sequence argument ([15, p. 86]), we only need to show that the inclusion above induces an isomorphism of the vertical deRham cohomology. This is true because the fiber of π is smoothly contractible.

Step 2:

Follow the usual double complex argument as in [2]. The augmented sequence of the horizontal complex is exact for every q :

$$0 \rightarrow (\Omega_{(L_2 \times G_1)}^q)^G \xrightarrow{\varepsilon_2} C_{\mathcal{R}}^0(G_2^\delta; \Omega_{(L_2 \times G_1)}^q) \xrightarrow{d_1} C_{\mathcal{R}}^1(G_2^\delta; \Omega_{(L_2 \times G_1)}^q) \xrightarrow{d_1} \dots$$

where $(\Omega_{(L_2 \times G_1)}^q)^{G_2}$ are G_2 -invariant forms on $L_2 \times G_1$, ε_2 is another inclusion. Because of this exactness we can use the usual diagram chasing argument for double complex with exact rows to get the fact that ε_2 induces an isomorphism from the cohomology of augmented column complex to the cohomology of double complex $C_{\mathcal{R}}^p(G_2^\delta; \Omega_{(L_2 \times G_1)}^q)$.

Step 3:

Finally the exactness of the above sequence is provided by the given homotopy (cf., [25, Lemma 2.3]):

$$(4.8) \quad \begin{aligned} H : C_{\mathcal{R}}^p(G_2^\delta; \Omega_{(L_2 \times G_1)}^q) &\rightarrow C_{\mathcal{R}}^{p-1}(G_2^\delta; \Omega_{(L_2 \times G_1)}^q) \\ H(\alpha)(\psi_1, \dots, \psi_p)(l \times m) &= (-1)^q \pi(f_{(\psi_1, \dots, \psi_p)}^\alpha), \end{aligned}$$

where

$$(4.9) \quad f_{(\psi_1, \dots, \psi_p)}^\alpha(h) := \alpha(hl, \psi_1, \dots, \psi_p)(l \times m),$$

and π the projection to its H_2 invariant part, since H_2 is reductive and every H_2 module is completely reducible. We have

$$(4.10) \quad \begin{aligned} & Hd_1(\alpha)(\psi_0, \dots, \psi_p)(l \times m) \\ &= (-1)^q \pi(f_{(\psi_0, \psi_1, \dots, \psi_p)}^{d_1 \alpha}) \\ &= \alpha(\psi_0, \dots, \psi_p)(l \times m) - \pi(f_{(\psi_1, \dots, \psi_p)}^\alpha) + \sum_{j=1}^p (-1)^{j+1} \pi(f_{(\psi_0, \dots, \check{\psi}_j, \dots, \psi_p)}^\alpha), \\ & d_1 H(\alpha)(\psi_0, \dots, \psi_p)(l \times m) = \sum_{j=0}^p (-1)^j \pi(f_{(\psi_0, \dots, \check{\psi}_j, \dots, \psi_p)}^\alpha). \end{aligned}$$

Therefore

$$(4.11) \quad Hd_1 + d_1H = I.$$

□

4.4. A smaller subcomplex $D^{p,q}$ and Connes' Φ map. The complex $C_{\mathcal{R}}^{p,q}$ has a subcomplex $D^{p,q}$, which consists of forms $\alpha(\psi_0, \dots, \psi_p)$ that can be written as

$$(4.12) \quad \alpha(\psi_0, \dots, \psi_p) = \sum c_j \rho_j,$$

where c_j are smooth functions on G_1 which are finite linear combinations of finite products of the following functions,

$$\varphi \in G_1 \rightarrow F_i((\psi_i \triangleleft \varphi)), \quad F_i \in \mathcal{R}(G_2),$$

more precisely,

$$(4.13) \quad c_j(\varphi) = \sum_{\text{finite}} \prod_{i=0}^p F_i((\psi_i \triangleleft \varphi)), \quad F_i \in \mathcal{R}(G_2).$$

The important fact about α is that it is not only G_2 -equivariant but also G -equivariant, i.e.,

$$(4.14) \quad \alpha(\psi_0 \triangleleft \phi, \dots, \psi_p \triangleleft \phi) = (\phi^{-1} \triangleright)^* \alpha(\psi_0, \dots, \psi_p), \quad \forall \psi_i \in G_2, \phi = \varphi \psi \in G_1 G_2 = G.$$

We use Connes' Φ map from bicomplex $(C^{p,q}, d_1, d_2)$ to the (b, B) bicomplex of the algebra $\mathcal{A} = C_c^\infty(G_1) \rtimes G_2^\delta$ and we show that the resulting cochains are representative as in 3.22, i.e., in the range of λ . Hence we have Φ from $(C^{p,q}, d_1, d_2)$ to the (b, B) bicomplex of \mathcal{H} . Let us recall the construction of Φ . As in [6], we let \mathcal{B} be the tensor product,

$$\mathcal{B} = A^*(G_1) \otimes \Lambda^*(\mathbb{C}(G'_2)),$$

where $A^*(G_1)$ is the algebra of smooth forms with compact support on G_1 , the exterior algebra $\Lambda^*(\mathbb{C}(G'_2))$ of the linear space $\mathbb{C}(G'_2)$ with basis the $\delta_\psi, \psi \in G_2$, with $\delta_e = 0$. We take the crossed product

$$\mathcal{C} = \mathcal{B} \rtimes G_2.$$

The action of G_2 on \mathcal{B} by automorphisms is defined as

$$(4.15) \quad \begin{aligned} U_\psi^* \omega U_\psi &= \tilde{\psi}^* \omega = \omega \circ \tilde{\psi} & \forall \omega \in A^*(G_1), \\ U_\psi^* \delta_{\psi_2} U_\psi &= \delta_{\psi_2 \psi} - \delta_\psi & \forall \psi_2 \in G_2. \end{aligned}$$

We write the elements of \mathcal{C} as a finite sum

$$c = \sum_{\psi} b U_\psi^*, \quad b \in \mathcal{B},$$

then we can write the differential d in \mathcal{C} as

$$(4.16) \quad d(b U_\psi^*) = db U_\psi^* - (-1)^{\partial b} b \delta_\psi U_\psi^*,$$

where the first term is only the exterior differential in $A^*(G_1)$.

A cochain γ in $C^{p,q}$ determines a form $\tilde{\gamma}$ on \mathcal{C} by

$$(4.17) \quad \tilde{\gamma}(\omega \otimes \delta_{\psi_1} \cdots \delta_{\psi_p}) = \int_{G_1} \omega \wedge \gamma(e, \psi_1, \dots, \psi_p), \quad \tilde{\gamma}(b U_\psi^*) = 0 \quad \text{if } \psi \neq e.$$

we define $\Phi(\gamma)$ by

$$(4.18) \quad \Phi(\gamma)(a^0, \dots, a^l) = \frac{p!}{(l+1)!} \sum_{j=0}^l (-1)^{j(l-j)} \tilde{\gamma}(da^{j+1} \dots da^l a^0 da^1 \dots da^j)$$

where $l = p + \dim(G_1) - q$, $a^0, \dots, a^l \in \mathcal{A}$.

We have the inclusions:

$$(4.19) \quad D^{p,q} \subset C_{\mathcal{R}}^{p,q} \subset C_d^{p,q} \subset C^{p,q}$$

and we use Φ_d , $\Phi_{\mathcal{R}}$, Φ_D to denote the restriction of Φ on these above subcomplexes, they are still chain maps because (b, B) maps will not change the type of the cochains.

Lemma 4.5. *Images of Φ_D are representative cochains as defined in 3.22.*

Proof. we follow exactly the same argument as in [7, p. 233-234]. When we need to rearrange the terms we use the coproduct of $\mathcal{R}(G_2)$, when we need to permute functions and U_{ψ}^* we use the fact that the left and right translates of representative function is still a representative function. \square

Proposition 4.6. Φ_D is a chain map from the bicomplex $(D^{p,q}, d_1, d_2)$ to the (b, B) -complex of the Hopf algebra \mathcal{H} .

Proof. It is proved in [6, III.2.8, Thm. 14] that Φ is a chain map from the bicomplex $(C^{p,q}, d_1, d_2)$ to the (b, B) -complex of the algebra \mathcal{A} . From the lemma above, the restriction Φ_D maps $D^{p,q}$ to representative cochain space, i.e., $Im(\lambda^{\natural})$, which is identified with $\mathcal{H}_{(\delta, \sigma)}^{\natural}$ as in Corollary 3.23. \square

5. EXPLICIT CONSTRUCTION OF HOPF CYCLIC CLASSES

As in diagram 1.1, the work of Rangipour and Sütliü ([34]) can be split into two steps: a vertical map from the Hopf cyclic complex of $\mathcal{R}(G_2) \blacktriangleleft \mathcal{U}(\mathfrak{g}_1)$ to the complex $C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*)$ and a horizontal map ν from $C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*)$ to the relative Lie algebra complex. The first process involves a number of quasi-isomorphic complexes and maps, hence will be recalled at the beginning of this section. We then construct an isomorphism Θ that link $C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*)$ with $D^{p,q}$ from previous section. In the middle of the section we construct an one-sided inverse of ν map, which we will call \mathcal{E} . To the end we prove the main theorem of our paper: $\mathcal{E} \circ \Theta \circ \mathcal{D}$ is a quasi-isomorphism, which goes along the opposite direction comparing to the maps of Rangipour and Sütliü, that can be used to transit cohomology classes from Lie algebra cohomology complex to representative cocycles on the convolution algebra explicitly.

5.1. Bicomplexes that calculate Hopf cyclic cohomology of \mathcal{H} . In this section we will review a few quasi-isomorphic bicomplex that are all quasi-isomorphic to the cyclic complex of \mathcal{H} . We refer to [31, 32, 34] for the detailed discussion. The first step is taken from [34], but with our setting; while the last three steps are extra steps from [31, 32], also with our setting, that help us to link to $D^{p,q}$.

step 1

From $CC^\bullet(\mathcal{H}, {}^\sigma \mathbb{C}_\delta)$ to $CC^\bullet(\mathcal{H}^{\text{cop}}, {}^{\sigma^{-1}} \mathbb{C}_\delta) = CC^\bullet(\mathcal{R}(G_2) \blacktriangleleft \mathcal{U}(\mathfrak{g}_1), {}^{\sigma^{-1}} \mathbb{C}_\delta)$.

It was shown in [31, Prop3.1] that

$$\begin{aligned} \mathcal{T} : CC^\bullet(\mathcal{H}, {}^\sigma \mathbb{C}_\delta) &\rightarrow CC^\bullet(\mathcal{H}^{\text{cop}}, {}^{\sigma^{-1}} \mathbb{C}_\delta), \\ \mathcal{T}(1 \otimes h^0 \otimes h^1 \times \cdots \otimes h^n) &= 1 \otimes \sigma^{-1} h^0 \otimes h^n \otimes \cdots \otimes h^1 \end{aligned}$$

defines an isomorphism of mixed complexes.

Recall that we have used antipode in our right coaction so the element σ in [34] is our σ^{-1} . It is proved in [34, 4.2] that our $CC^\bullet(\mathcal{R}(G_2) \blacktriangleleft \mathcal{U}(\mathfrak{g}_1), {}^{\sigma^{-1}} \mathbb{C}_\delta)$ is quasi-isomorphic to a double complex

$$(5.1) \quad \begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & \\ \wedge^2 \mathfrak{g}_1^* & \xleftarrow[b_{\mathcal{R}}]{B_{\mathcal{R}}} & \wedge^2 \mathfrak{g}_1^* \otimes \mathcal{R} & \xleftarrow[b_{\mathcal{R}}]{B_{\mathcal{R}}} & \wedge^2 \mathfrak{g}_1^* \otimes \mathcal{R}^{\otimes 2} & \xleftarrow[b_{\mathcal{R}}]{B_{\mathcal{R}}} & \cdots \\ & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & \\ \mathfrak{g}_1^* & \xleftarrow[b_{\mathcal{R}}]{B_{\mathcal{R}}} & \mathfrak{g}_1^* \otimes \mathcal{R} & \xleftarrow[b_{\mathcal{R}}]{B_{\mathcal{R}}} & \mathfrak{g}_1^* \otimes \mathcal{R}^{\otimes 2} & \xleftarrow[b_{\mathcal{R}}]{B_{\mathcal{R}}} & \cdots \\ & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & \\ \mathbb{C} & \xleftarrow[b_{\mathcal{R}}]{B_{\mathcal{R}}} & \mathbb{C} \otimes \mathcal{R} & \xleftarrow[b_{\mathcal{R}}]{B_{\mathcal{R}}} & \mathbb{C} \otimes \mathcal{R}^{\otimes 2} & \xleftarrow[b_{\mathcal{R}}]{B_{\mathcal{R}}} & \cdots \end{array}$$

the coboundary $\partial_{\mathfrak{g}}$ is the Lie algebra cohomology coboundary of \mathfrak{g}_1 with coefficients in $\mathcal{R}^{\otimes *}$ with right \mathfrak{g}_1 action given by

$$(5.2) \quad (F^1 \otimes \cdots \otimes F^p) \triangleleft Z = -Z \triangleright (F^1 \otimes \cdots \otimes F^p),$$

here the left action of \mathfrak{g}_1 on $\mathcal{R}^{\otimes p}$ is extended from 3.17 but is different from the usual diagonal action. Because the coproduct of \mathfrak{g}_1 we used here is not from $\mathcal{U}(\mathfrak{g}_1)$ but from $\mathcal{R}(G_2) \blacktriangleleft \mathcal{U}(\mathfrak{g}_1)$. The formula is given by

$$\begin{aligned} Z \triangleright (F^1 \otimes \cdots \otimes F^p) &= \\ Z_{(1) < 0 >} \triangleright F^1 \otimes Z_{(1) < 1 >} (Z_{(2) < 0 >} \triangleright F^2) \otimes \cdots \otimes Z_{(1) < p-1 >} \cdots Z_{(p-1) < 1 >} (Z_{(p)} \triangleright F^p). \end{aligned}$$

The coboundary $b_{\mathcal{R}}$ involve the coaction \blacktriangledown . It is the b operator for coalgebra \mathcal{R} with coefficients in $\wedge^\bullet \mathfrak{g}_1^*$. The left coaction is given by 3.16

$$\blacktriangledown(\omega^i) = S(\gamma_j^i) \otimes \omega^j,$$

and then extend to $\wedge^\bullet \mathfrak{g}_1^*$.

It has the explicit expression

$$(5.3) \quad \begin{aligned} b_{\mathcal{R}}(\alpha \otimes F^1 \otimes \cdots \otimes F^p) &= \alpha \otimes 1 \otimes F^1 \otimes \cdots \otimes F^p \\ &+ \sum_{1 \leq i \leq p} (-1)^i \alpha \otimes F^1 \otimes \cdots \otimes \Delta(F^i) \otimes \cdots \otimes F^p \\ &+ (-1)^{p+1} \alpha_{< 0 >} \otimes F^1 \otimes \cdots \otimes F^p \otimes \alpha_{< -1 >} \end{aligned}$$

$$\begin{aligned}
(5.4) \quad B_{\mathcal{R}} &= \left(\sum_{i=0}^{p-1} (-1)^{(p-1)i} \tau_{\mathcal{R}}^i \right) \sigma_{\mathcal{R}} \tau_{\mathcal{R}} (1 - (-1)^p \tau_{\mathcal{R}}), \quad \text{with} \\
\tau_{\mathcal{R}}(\alpha \otimes F^1 \otimes \cdots \otimes F^p) &= \alpha_{<0>} \otimes S(F^1) \cdot (F^2 \otimes \cdots \otimes F^p \otimes \alpha_{<-1>}) \\
\sigma_{\mathcal{R}}(\alpha \otimes F^1 \otimes \cdots \otimes F^p) &= \varepsilon(F^p) \alpha \otimes F^1 \otimes \cdots \otimes F^{p-1}.
\end{aligned}$$

Here we see no σ^{-1} or δ , because during the transition we have identified the module $\sigma^{-1}\mathbb{C}_{\delta}$ with $\wedge^{\dim(\mathfrak{g}_1)}\mathfrak{g}_1^*$ via Poincaré isomorphism.

In [34], the authors stopped at this complex and derived the desired quasi-isomorphism between relative Lie algebra cohomology and the Hopf cyclic cohomology that we begin from. However, we want to go to $C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*)$ and $D^{p,q}$. Therefore we pass from the previous complex to its homogeneous version and then to the equivariant cochains.

step 2

homogeneous version

We define $C_{\mathcal{R}}^{p,q}(\wedge^q \mathfrak{g}_1^* \otimes \mathcal{R}(G_2)) := (\wedge^q \mathfrak{g}_1^* \otimes \mathcal{R}(G_2)^{\otimes p+1})^{\mathcal{R}(G_2)}$. An element $\sum \alpha \otimes F^0 \otimes \cdots \otimes F^p$ is in $(\wedge^p \mathfrak{g}_1^* \otimes \mathcal{R}(G_2)^{\otimes p+1})^{\mathcal{R}(G_2)}$ if it satisfies the \mathcal{R} -coinvariance condition:

$$\sum \alpha_{<0>} \otimes F^0 \otimes \cdots \otimes F^p \otimes \alpha_{<-1>} = \sum \alpha \otimes F_{(1)}^0 \otimes F_{(1)}^1 \otimes \cdots \otimes F_{(1)}^p \otimes F_{(2)}^0 \cdots F_{(2)}^p.$$

The two complexes are isomorphic via

$$\begin{aligned}
(5.5) \quad \mathcal{I} : \wedge^q \mathfrak{g}_1^* \otimes \mathcal{R}(G_2)^{\otimes p} &\rightarrow (\wedge^q \mathfrak{g}_1^* \otimes \mathcal{R}(G_2)^{\otimes p+1})^{\mathcal{R}(G_2)}, \\
\mathcal{I}(\alpha \otimes F^1 \otimes \cdots \otimes F^p) &= \\
\alpha_{<0>} \otimes F_{(1)}^1 \otimes S(F_{(2)}^1) F_{(1)}^2 \otimes \cdots \otimes S(F_{(2)}^{p-1}) F_{(1)}^p \otimes S(F_{(2)}^p) \alpha_{<-1>} &;
\end{aligned}$$

and

$$\begin{aligned}
(5.6) \quad \mathcal{I}^{-1} : (\wedge^q \mathfrak{g}_1^* \otimes \mathcal{R}(G_2)^{\otimes p+1})^{\mathcal{R}(G_2)} &\rightarrow \wedge^q \mathfrak{g}_1^* \otimes \mathcal{R}(G_2)^{\otimes p}, \\
\mathcal{I}^{-1}(\alpha \otimes F^0 \otimes \cdots \otimes F^p) &= \\
\alpha \otimes F_{(1)}^0 \otimes F_{(2)}^0 F_{(1)}^1 \otimes F_{(3)}^0 F_{(2)}^1 F_{(1)}^2 \otimes \cdots \otimes F_{(p)}^0 \cdots F_{(2)}^{p-2} F_{(1)}^{p-1} \varepsilon(F^p). &
\end{aligned}$$

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} \\
\wedge^2 \mathfrak{g}_1^* & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & (\wedge^2 \mathfrak{g}_1^* \otimes \mathcal{R}^{\otimes 2})^{\mathcal{R}} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & (\wedge^2 \mathfrak{g}_1^* \otimes \mathcal{R}^{\otimes 3})^{\mathcal{R}} \xrightarrow{b_{\mathcal{R}}} \dots \\
\uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} \\
\mathfrak{g}_1^* & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & (\wedge^1 \mathfrak{g}_1^* \otimes \mathcal{R}^{\otimes 2})^{\mathcal{R}} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & (\wedge^1 \mathfrak{g}_1^* \otimes \mathcal{R}^{\otimes 3})^{\mathcal{R}} \xrightarrow{b_{\mathcal{R}}} \dots \\
\uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} \\
\mathbb{C} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & (\mathbb{C} \otimes \mathcal{R}^{\otimes 2})^{\mathcal{R}} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & (\mathbb{C} \otimes \mathcal{R}^{\otimes 3})^{\mathcal{R}} \xrightarrow{b_{\mathcal{R}}} \dots
\end{array}
\tag{5.7}$$

We transform these boundary operators via \mathcal{I} and get the coboundary $\partial_{\mathfrak{g}_1}$ is the Lie algebra cohomology coboundary of \mathfrak{g}_1 with coefficients in $\mathcal{R}^{\otimes \bullet}$ with right \mathfrak{g}_1 action now given by the usual diagonal action:

$$\begin{aligned}
& (1 \otimes F^1 \otimes \dots \otimes F^p) \triangleleft Z \\
& = -1 \otimes \left(\sum_{i=1}^p F^1 \otimes \dots \otimes Z \triangleright F^i \otimes \dots \otimes F^p \right)
\end{aligned}$$

while $b_{\mathcal{R}}$ has a simple expression,

$$(5.8) \quad b_{\mathcal{R}}(\alpha \otimes F^0 \otimes \dots \otimes F^p) = \sum_{i=0}^{p+1} (-1)^i \alpha \otimes F^0 \otimes \dots \otimes F^{i-1} \otimes 1 \otimes F^i \otimes \dots \otimes F^p.$$

$B_{\mathcal{R}}$ also has a simple expression,

$$(5.9) \quad B_{\mathcal{R}} = \left(\sum_{i=0}^{p-1} (-1)^{(p-1)i} \tau_{\mathcal{R}}^i \right) \sigma_{\mathcal{R}} \tau_{\mathcal{R}} (1 - (-1)^p \tau_{\mathcal{R}}),$$

with

$$\begin{aligned}
(5.10) \quad & \tau_{\mathcal{R}}(\alpha \otimes F^0 \otimes \dots \otimes F^p) = \alpha \otimes F^1 \otimes \dots \otimes F^p \otimes F^0 \\
& \sigma_{\mathcal{R}}(\alpha \otimes F^0 \otimes \dots \otimes F^p) = \alpha \otimes F^0 \otimes \dots \otimes F^{p-1} F^p.
\end{aligned}$$

step 3

Usually we will look at the complex of totally antisymmetric cochains $C_{\mathcal{R}}^{p,q}(\wedge \mathfrak{g}_1^*, \wedge \mathcal{R}(G_2)) := (\wedge^q \mathfrak{g}_1^* \otimes \wedge^{p+1} \mathcal{R}(G_2))^{\mathcal{R}(G_2)}$

It is quasi-isomorphic to $C_{\mathcal{R}}^{p,q}(\wedge \mathfrak{g}_1^*, \otimes \mathcal{R}(G_2))$ via the antisymmetrization map $\alpha_{\mathcal{R}}$

$$\alpha_{\mathcal{R}}(\alpha \otimes F^0 \wedge \dots \wedge F^p) = \frac{1}{(p+1)!} \sum_{\sigma \in S_{p+1}} (-1)^{\sigma} \alpha \otimes F^{\sigma(0)} \otimes \dots \otimes F^{\sigma(p)}.$$

The quasi-isomorphism is the same as in the case of group cohomology. We refer to [41, Appendix A].

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \\
 \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \\
 \wedge^2 \mathfrak{g}_1^* & \xrightarrow{b_{\mathcal{R}}} & (\wedge^2 \mathfrak{g}_1^* \otimes \wedge^2 \mathcal{R})^{\mathcal{R}} & \xrightarrow{b_{\mathcal{R}}} & (\wedge^2 \mathfrak{g}_1^* \otimes \wedge^3 \mathcal{R})^{\mathcal{R}} & \xrightarrow{b_{\mathcal{R}}} & \dots \\
 \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \\
 \mathfrak{g}_1^* & \xrightarrow{b_{\mathcal{R}}} & (\wedge^1 \mathfrak{g}_1^* \otimes \wedge^2 \mathcal{R})^{\mathcal{R}} & \xrightarrow{b_{\mathcal{R}}} & (\wedge^1 \mathfrak{g}_1^* \otimes \wedge^3 \mathcal{R})^{\mathcal{R}} & \xrightarrow{b_{\mathcal{R}}} & \dots \\
 \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \\
 \mathbb{C} & \xrightarrow{b_{\mathcal{R}}} & (\mathbb{C} \otimes \wedge^2 \mathcal{R})^{\mathcal{R}} & \xrightarrow{b_{\mathcal{R}}} & (\mathbb{C} \otimes \wedge^3 \mathcal{R})^{\mathcal{R}} & \xrightarrow{b_{\mathcal{R}}} & \dots
 \end{array}$$

The boundary operators have simple expression when restricted to such complex.

$$\begin{aligned}
 b_{\mathcal{R}}(\alpha \otimes F^0 \wedge \dots \wedge F^p) &= \alpha \otimes 1 \wedge F^0 \wedge \dots \wedge F^p, \\
 \partial_{\mathfrak{g}}(\alpha \otimes F^0 \wedge \dots \wedge F^p) &= \partial \alpha \otimes F^0 \wedge \dots \wedge F^p \\
 &\quad - \sum_{i,j} \omega^i \wedge \alpha \otimes F^0 \wedge \dots \wedge Z_i \triangleright F^j \wedge \dots \wedge F^p,
 \end{aligned}
 \tag{5.11}$$

and because

$$\tau_{\mathcal{R}}(\alpha \otimes F^0 \wedge \dots \wedge F^p) = \alpha \otimes F^1 \wedge \dots \wedge F^p \wedge F^0 = (-1)^p \alpha \otimes F^0 \wedge \dots \wedge F^p,$$

we have

$$(1 - (-1)^p \tau_{\mathcal{R}})(\alpha \otimes F^0 \wedge \dots \wedge F^p) = 0,$$

therefore

$$B_{\mathcal{R}}(\alpha \otimes F^0 \wedge \dots \wedge F^p) = 0.$$

We notice that $C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*)$ is quasi-isomorphic to $C_{\mathcal{R}}^{p,q}(\wedge \mathfrak{g}_1^*, \wedge \mathcal{R}(G_2))$ via the map:

$$\mathcal{J}(\alpha \otimes F^0 \wedge \dots \wedge F^p)(\psi_0, \dots, \psi_p) = \frac{1}{(p+1)!} \sum_{\sigma \in S_{p+1}} (-1)^\sigma \alpha F^{\sigma(0)}(\psi_0) \dots F^{\sigma(p)}(\psi_p).$$

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & \\
& \wedge^2 \mathfrak{g}_1^* & \xrightarrow{b_{\mathcal{R}}} & C_{\mathcal{R}}^1(G_2, \wedge^2 \mathfrak{g}_1^*) & \xrightarrow{b_{\mathcal{R}}} & C_{\mathcal{R}}^2(G_2, \wedge^2 \mathfrak{g}_1^*) & \xrightarrow{b_{\mathcal{R}}} \dots \\
& \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & \\
& \mathfrak{g}_1^* & \xrightarrow{b_{\mathcal{R}}} & C_{\mathcal{R}}^1(G_2, \wedge^1 \mathfrak{g}_1^*) & \xrightarrow{b_{\mathcal{R}}} & C_{\mathcal{R}}^2(G_2, \wedge^1 \mathfrak{g}_1^*) & \xrightarrow{b_{\mathcal{R}}} \dots \\
& \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & & \uparrow \partial_{\mathfrak{g}} & \\
\mathbb{C} & \xrightarrow{b_{\mathcal{R}}} & C_{\mathcal{R}}^1(G_2, \mathbb{C}) & \xrightarrow{b_{\mathcal{R}}} & C_{\mathcal{R}}^2(G_2, \mathbb{C}) & \xrightarrow{b_{\mathcal{R}}} \dots
\end{array}
\tag{5.12}$$

The boundary operators are

$$\begin{aligned}
b_{\mathcal{R}}(c)(\psi_0, \dots, \psi_{p+1}) &= \sum_{i=0}^{p+1} (-1)^i c(\psi_0, \dots, \hat{\psi}_i, \dots, \psi_{p+1}) \\
\partial_{\mathfrak{g}}(c)(\psi_0, \dots, \psi_p) &= \partial(c(\psi_0, \dots, \psi_p)) - \sum_i \omega^i \wedge (Z_i \triangleright c)(\psi_0, \dots, \psi_p).
\end{aligned}$$

step 4

we define a map $\Theta : C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*) \rightarrow D^{p,q}$

$$\Theta(\alpha)(\psi_0, \dots, \psi_p)|_{\varphi} = L_{\varphi}^*(\alpha)(\psi_0 \triangleleft \varphi, \dots, \psi_p \triangleleft \varphi)
\tag{5.13}$$

Proposition 5.1. Θ is an isomorphism between complexes.

Proof. use the strong covariance property of $D^{p,q}$, cochains in $D^{p,q}$ depend only on its value at $e_1 \in G_1$. \square

5.2. Construction from Lie algebra cohomology complex to $D^{p,q}$. In this section, we would like to carry out a parallel construction of a chain map from Lie algebra cohomology complex of the pair $(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{h}_2)$ to the complex $D^{p,q}$, comparing with the infinite dimensional case as in [32, 2.1], and [7].

Assume H_2 is invariant under G_1 action, we can have $H_2 \setminus (G_1 \bowtie G_2) \cong G_1 \bowtie (H_2 \setminus G_2)$ and $\mathfrak{h}_2 \setminus (\mathfrak{g}_1 \bowtie \mathfrak{g}_2) \cong \mathfrak{g}_1 \bowtie (\mathfrak{h}_2 \setminus \mathfrak{g}_2)$. We can also write down right \mathfrak{h}_2 -action on $\mathfrak{h}_2 \setminus (\mathfrak{g}_1 \bowtie \mathfrak{g}_2) \cong \mathfrak{g}_1 \bowtie (\mathfrak{h}_2 \setminus \mathfrak{g}_2)$ by the induced adjoint action $(-\triangleright) \oplus ad$. In order to have a linear action we need to assume that the action \triangleright of \mathfrak{h}_2 on \mathfrak{g}_1 is given by derivations, this is the case when our matched pair of Lie groups are from decomposition of Lie group. Under above assumptions we can talk about \mathfrak{h}_2 -basic forms on the bicrossed product group $G_1 \bowtie G_2$.

For $\psi_0, \dots, \psi_p \in G_2$, we let $\Delta(\psi_0, \dots, \psi_p)$ be the affine simplex with vertices $\pi_L(\psi_i)$ in the affine coordinates on L_2 . i.e.,

$$\Delta(\psi_0, \dots, \psi_p) = \exp \left(\sum_0^p t_i \log (\pi_L(\psi_i)) \right); \quad 0 \leq t_i \leq 1, \sum_0^p t_i = 1$$

where π_L is the projection $G_2 \rightarrow H_2 \backslash G_2$. We check that the right multiplication of G_2 is affine on the exponential coordinates of L_2 , i.e., assume $\psi = hl$,

$$\begin{aligned}
 \Delta(\psi_0\psi, \dots, \psi_p\psi) &= \exp \left(\sum_0^p t_i \log (\pi_L(\psi_i\psi)) \right) \\
 (5.14) \qquad &= \exp \left(\sum_0^p t_i \log ((\pi_L(\psi_i) \triangleleft h)l) \right) \\
 &= (\Delta(\psi_0, \dots, \psi_p) \triangleleft h)l
 \end{aligned}$$

where we identify L_2 with $H_2 \backslash G_2$ to make sense of the right G_2 action.

We assume that the right G_1 action is affine on the exponential coordinates, i.e.,

$$\begin{aligned}
 \Delta(\psi_0 \triangleleft \varphi, \dots, \psi_p \triangleleft \varphi) &= \exp \left(\sum_0^p t_i \log (\pi_L(\psi_i \triangleleft \varphi)) \right) \\
 (5.15) \qquad &= \exp \left(\sum_0^p t_i \log (\pi_L(\psi_i) \triangleleft \varphi) \right) \\
 &= \Delta(\psi_0, \dots, \psi_p) \triangleleft \varphi
 \end{aligned}$$

We define \mathcal{D} as discussed in [32, 2.1] and [7, p229].

$$(5.16) \qquad \mathcal{D} : (\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2} \rightarrow C^{p,q}$$

$$(5.17) \qquad \langle \mathcal{D}(\omega)(\psi_0, \dots, \psi_p), \eta \rangle = (-1)^{\frac{q(q+1)}{2}} \int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} \pi_1^* \eta \wedge \tilde{\omega}$$

Lemma 5.2. *The group cochain $\mathcal{D}(\omega)$ satisfies the strong covariance property:*

$$(5.18) \qquad \mathcal{D}(\omega)(\psi_0 \triangleleft \phi, \dots, \psi_p \triangleleft \phi) = (\phi^{-1} \triangleright)^* \mathcal{D}(\omega)$$

where $\phi = \varphi\psi \in G_1 G_2 = G$.

Proof.

$$\begin{aligned}
 \langle \mathcal{D}(\omega)(\psi_0\phi, \dots, \psi_p\phi), \eta \rangle &= \int_{(G_1 \times \Delta(\psi_0\phi, \dots, \psi_p\phi))^{-1}} \pi_1^* \eta \wedge \tilde{\omega} \\
 &= \int_{L_{\phi^{-1}}(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} \pi_1^* \eta \wedge \tilde{\omega} \\
 &= \int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} (\tilde{\phi})^* (\pi_1^* \eta \wedge \tilde{\omega}) \\
 &= \int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} \pi_1^* ((\phi \triangleright)^* \eta) \wedge \tilde{\omega} \\
 &= \langle \mathcal{D}(\omega)(\psi_0, \dots, \psi_p), (\phi \triangleright)^* \eta \rangle \\
 &= \langle (\phi^{-1} \triangleright)^* \mathcal{D}(\omega)(\psi_0, \dots, \psi_p), \eta \rangle
 \end{aligned}$$

□

Proposition 5.3. *\mathcal{D} is a map of complexes*

Proof. The covariance property is checked above. Now we check

$$(5.19) \quad \mathcal{D}(d\omega) = (d_1 + d_2)\mathcal{D}(\omega)$$

we have

$$\begin{aligned}
 (5.20) \quad & \langle \mathcal{D}(d\omega)(\psi_0, \dots, \psi_p), \eta \rangle \\
 &= (-1)^{\frac{q(q+1)}{2}} \int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} \pi_1^* \eta \wedge d\tilde{\omega} \\
 &= (-1)^{\frac{q(q+1)}{2}} (-1)^q \left(\int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} d(\pi_1^* \eta \wedge \tilde{\omega}) \right. \\
 &\quad \left. - \int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} \pi_1^*(d\eta) \wedge \tilde{\omega} \right) \\
 &= (-1)^q (-1)^{\frac{q(q+1)}{2}} \left(\int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} d(\pi_1^* \eta \wedge \tilde{\omega}) \right) \\
 &\quad + (-1)^{\frac{(q+1)(q+2)}{2}} \left(\int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} \pi_1^*(d\eta) \wedge \tilde{\omega} \right) \\
 &= (-1)^q (-1)^{\frac{q(q+1)}{2}} \left(\int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} d(\pi_1^* \eta \wedge \tilde{\omega}) \right) + \langle d_2 \mathcal{D}(\omega), \eta \rangle.
 \end{aligned}$$

Apply Stokes, the first integral becomes the group coboundary:

$$\begin{aligned}
 (5.21) \quad & (-1)^q (-1)^{\frac{q(q+1)}{2}} \left(\int_{(G_1 \times \Delta(\psi_0, \dots, \psi_p))^{-1}} d(\pi_1^* \eta \wedge \tilde{\omega}) \right) \\
 &= (-1)^q (-1)^{\frac{q(q+1)}{2}} \left(\sum_{i=0}^{p+1} (-1)^i \int_{(G_1 \times \Delta(\psi_0, \dots, \hat{\psi}_i, \dots, \psi_p))^{-1}} \pi_1^* \eta \wedge \tilde{\omega} \right) \\
 &= \sum_{i=0}^{p+1} (-1)^{q+i} \langle \mathcal{D}(\omega)(\psi_0, \dots, \hat{\psi}_i, \dots, \psi_p), \eta \rangle \\
 &= \langle d_1 \mathcal{D}(\omega), \eta \rangle.
 \end{aligned}$$

□

Because of the strong covariance property, D is completely determined by its value at $e_1 \in G_1$, consider group cochains with values in $\wedge^\bullet \mathfrak{g}_1^*$:

$$(5.22) \quad \mathcal{E}(\omega)(\psi_0, \dots, \psi_p) := \mathcal{D}(\omega)(\psi_0, \dots, \psi_p)|_{e_1}.$$

Corollary 5.4. \mathcal{E} is a chain map from $(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2}$ to $C^p(G_2, \wedge^q \mathfrak{g}_1^*)$

Proof. The covariance property is preserved by the evaluation.

Use Prop. 5.3, the evaluation of the second integral in its proof at e_1 is clearly group coboundary, while the first integral, when evaluate at e_1 , gives Lie algebra coboundary with coefficients. □

As discussed in [32, 2.1] we can also verify the explicit description of \mathcal{E} : Fix a basis Z_i of \mathfrak{g}_1 , denote by \tilde{Z}_i the corresponding left invariant vector fields, ω_i the dual basis, $\tilde{\omega}_i$ the corresponding left invariant forms fields. Use the formula in [32, 2.2], we define $\nu(\varphi, \psi) = \varphi(\psi \lrcorner \varphi)^{-1}$. Let $\iota : L_2 \rightarrow L_2$ be the inversion map, ι be the contraction. Let ω be

in $(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2}$ and $\tilde{\omega}$ the corresponding left invariant \mathfrak{h}_2 -basic form on $(G_1 \ltimes G_2)/H_2$. It is the key that we can have explicit expression of such invariant form in order to make use of this explicit \mathcal{E} map. Denote

$$(5.23) \quad \begin{aligned} \mu : (\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2} &\rightarrow \sum_{p+q=n} (\Omega^p(L_2))^{\mathfrak{h}_2} \otimes \wedge^q \mathfrak{g}_1^* \\ \mu_q(\omega) &= \sum_{|I|=q} \iota^*(\iota_{\tilde{Z}_I(e)} \nu^*(\tilde{\omega})) \otimes \omega_I, \\ I &= (i_1 < \dots < i_q) \quad \text{and} \quad \omega_I = \omega_{i_1} \wedge \dots \wedge \omega_{i_q} \end{aligned}$$

Use the formula in [32, 2.2] We then write a map $\mathcal{E} = \int_{\Delta} \circ \mu$ from $(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2}$ to $\sum_{p+q=n} C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*)$:

$$(5.24) \quad \mathcal{E}(\omega)(\psi_0, \dots, \psi_p) = \int_{\Delta(\psi_0, \dots, \psi_p)} \mu_q(\omega)$$

one can also compare this map with the integration along simplex map as used by Dupont in [12]. Then we can have

$$(5.25) \quad \mathcal{E}(\omega)(\psi_0, \dots, \psi_p) = \mathcal{D}(\omega)(\psi_0, \dots, \psi_p)|_{e_1}$$

Proposition 5.5. $\mathcal{E}(\omega)$ are representative functions of ψ_i s.

Proof. Because $L_2 = \exp(\mathfrak{l}_2)$ is nilpotent, representative functions are exactly the polynomial functions. Use exponential coordinates on L_2 and we can represent the projection of $\nu^*(\tilde{\omega})$ on L_2 as forms with polynomial coefficients. Now apply the contractions $\iota_{\tilde{Z}_I}$, the only new type of coefficients that appear are of the form $Z_I \triangleright P$ where P is some polynomial function of L_2 . By Lemma 3.2 $Z_I \triangleright P$ is again representative hence polynomial. Therefore $\iota^*(\iota_{\tilde{Z}_I(e)} \nu^*(\tilde{\omega}))$ are polynomial differential forms on L_2 and $\mathcal{E}(\omega)$ are representative functions on L_2 .

Furthermore they are H_2 invariant, every H_2 invariant L_2 representative function is representative on G_2 because of the semi direct product decomposition $G_2 = H_2 \ltimes L_2$. More precisely if we extend any $F \in \mathcal{R}(L_2)$ trivially to a function on G_2 by $F(hl) = F(l)$, then

$$\begin{aligned} F((h_1 l_1)(h_2 l_2)) &= F((h_1 h_2)(l_1 \triangleleft h_2 l_2)) = F(l_1 \triangleleft h_2 l_2) = F_{(1)}(l_1 \triangleleft h_2) F_{(2)}(l_2) \\ &= F_{(1)}(l_1) F_{(2)}(l_2) = F_{(1)}(h_1 l_1) F_{(2)}(h_2 l_2), \end{aligned}$$

hence $F \in \mathcal{R}(G_2)$ and π_L^* preserves representative functions.

Therefore $\mathcal{E}(\omega)$ are representative functions on G_2 . □

Corollary 5.6. \mathcal{E} is a chain map from $(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2}$ to $C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*)$

Proof. Use previous proposition and corollary 5.4. □

Corollary 5.7. \mathcal{D} is a chain map from $(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2}$ to $D^{p,q} \subset C^{p,q}$

Proof. The map \mathcal{E} can be extended by the isomorphism $\Theta : C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*) \rightarrow D^{p,q}$ defined in 5.13

$$\Theta(\alpha)(\psi_0, \dots, \psi_p)|_{\varphi} = L_{\varphi}^*(\alpha)(\psi_0 \triangleleft \varphi, \dots, \psi_p \triangleleft \varphi)$$

back to map $\mathcal{D} = \Theta \circ \mathcal{E}$:

$$\mathcal{D}(\omega)(\psi_0, \dots, \psi_p)|_\varphi = L_\varphi^*(\mathcal{E}(\omega))(\psi_0 \triangleleft \varphi, \dots, \psi_p \triangleleft \varphi)$$

□

We can also have the identification below in this case: $(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2} \cong \bigoplus_{p+q=n} (\wedge^q \mathfrak{g}_1^* \otimes \wedge^p(\mathfrak{l}_2^*)^{\mathfrak{h}_2})$ follow the proof in [32, Lemma 2.7] the isomorphism is given by

$$(5.26) \quad \begin{aligned} \mathfrak{h} : (\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2} &\rightarrow \bigoplus_{p+q=n} (\wedge^q \mathfrak{g}_1^* \otimes \wedge^p(\mathfrak{l}_2^*)^{\mathfrak{h}_2}) \\ \mathfrak{h}(\omega)(X^1, \dots, X^q | \xi_1, \dots, \xi_p) &= \omega(X^1 \oplus 0, \dots, X^q \oplus 0, 0 \oplus \xi_1, \dots, 0 \oplus \xi_p) \end{aligned}$$

with its inverse

$$(5.27) \quad \begin{aligned} &\mathfrak{h}^{-1}(\mu \otimes \nu)(X^1 \oplus \xi_1, \dots, X^{p+q} \oplus \xi_{p+q}) \\ &= \sum_{\sigma \in Sh(p,q)} (-1)^\sigma \mu(X^{\sigma(1)}, \dots, X^{\sigma(p)}) \nu(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}) \end{aligned}$$

Use a slight modification of [32, Lemma. 2.8, Cor. 2.9] we can show that $\mu|_{e_2}$ coincides with \mathfrak{h} .

For $\xi_1, \dots, \xi_p \in \mathfrak{l}_2$, denote

$$\begin{aligned} \Psi(\xi_1, \dots, \xi_p) &= \\ &(1, \exp(s_{\sigma(1)} \xi_{\sigma(1)}), \exp(s_{\sigma(1)} \xi_{\sigma(1)}) \exp(s_{\sigma(2)} \xi_{\sigma(2)}), \dots, \exp(s_{\sigma(1)} \xi_{\sigma(1)}) \cdots \exp(s_{\sigma(p)} \xi_{\sigma(p)})) \end{aligned}$$

Define a map $j : C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*) \rightarrow \wedge^q \mathfrak{g}_1^* \otimes \wedge^p(\mathfrak{l}_2^*)^{\mathfrak{h}_2}$

$$\begin{aligned} &j(\alpha)(X^1, \dots, X^q | \xi_1, \dots, \xi_p) \\ &= \sum_{\sigma \in S_p} (-1)^\sigma \left. \frac{d}{ds_1} \right|_{s_1=0} \cdots \left. \frac{d}{ds_p} \right|_{s_p=0} \alpha(\Psi(\xi_1, \dots, \xi_p))(X^1, \dots, X^q) \end{aligned}$$

then The map \mathcal{E} has a quasi-inverse $\mathfrak{h}^{-1} \circ j$.

Lemma 5.8. $(\mathfrak{h}^{-1} \circ j) \circ \mathcal{E}$ is identity.

Proof. since $\mu|_{e_2}$ implement the \mathfrak{h} map, we just need to show that $j \circ \int_{\Delta}$ is the evaluation at e_2 . we write $\xi_k = \xi_k^j \frac{\partial}{\partial x_j}$ and write any p form as $f_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ in the exponential coordinates.

$$\begin{aligned} x^{i_k} &= t_1(s_{\sigma(1)} \xi_{\sigma(1)}^{i_k}) + t_2(s_{\sigma(1)} \xi_{\sigma(1)}^{i_k} + s_{\sigma(2)} \xi_{\sigma(2)}^{i_k} + p_2(s_{\sigma(1)}, s_{\sigma(2)})) + \cdots \\ &\quad + t_k(s_{\sigma(1)} \xi_{\sigma(1)}^{i_k} + \cdots + s_{\sigma(k)} \xi_{\sigma(k)}^{i_k} + p_{\sigma(k)}(s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(k)})), \quad 1 \leq k \leq p \end{aligned}$$

where p_2, \dots, p_k are ξ -valued polynomials with degree higher than 2.

$$\begin{aligned} d(x^{i_k}) &= (s_{\sigma(1)} \xi_{\sigma(1)}^{i_k}) dt_1 + (s_{\sigma(1)} \xi_{\sigma(1)}^{i_k} + s_{\sigma(2)} \xi_{\sigma(2)}^{i_k} + p_2(s_{\sigma(1)}, s_{\sigma(2)})) dt_2 + \cdots \\ &\quad + (s_{\sigma(1)} \xi_{\sigma(1)}^{i_k} + \cdots + s_{\sigma(k)} \xi_{\sigma(k)}^{i_k} + p_{\sigma(k)}(s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(k)})) dt_k, \quad 1 \leq k \leq p \end{aligned}$$

therefore if we calculate the wedge product of $dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, its degree will be at least p , viewed as $\xi dt^1 \wedge \cdots \wedge dt^p$ -valued polynomials in s_j . Since we want to take derivative from s_1 to s_p , there is exact one term that gives non-zero value:

$$\begin{aligned}
& \sum_{\sigma \in S_p} (-1)^\sigma \frac{d}{ds_1} \Big|_{s_1=0} \cdots \frac{d}{ds_p} \Big|_{s_p=0} \\
& \int_{\Delta(\Psi(\xi_1, \dots, \xi_p))} \sum_{i_1 < \dots < i_p} f_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \\
& = \sum_{\sigma \in S_p} (-1)^\sigma \frac{d}{ds_1} \Big|_{s_1=0} \cdots \frac{d}{ds_p} \Big|_{s_p=0} \\
& \int_{\Delta^p} \sum_{i_1 < \dots < i_p} f_{i_1 \dots i_p} s_{\sigma(1)} \xi_{\sigma(1)}^{i_1} dt_1 \wedge \dots \wedge s_{\sigma(p)} \xi_{\sigma(p)}^{i_p} dt_p \\
& = \sum_{\sigma \in S_p} (-1)^\sigma \int_{\Delta^p} \sum_{i_1 < \dots < i_p} f_{i_1 \dots i_p}(e_2) \xi_{\sigma(1)}^{i_1}(e_2) dt_1 \wedge \dots \wedge \xi_{\sigma(p)}^{i_p}(e_2) dt_p \\
& = \sum_{\sigma \in S_p} (-1)^\sigma \frac{1}{p!} \sum_{i_1 < \dots < i_p} (f_{i_1 \dots i_p} \xi_{\sigma(1)}^{i_1} \cdots \xi_{\sigma(p)}^{i_p})(e_2) \\
& = (f_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}) (\xi_1^j \frac{\partial}{\partial x_j}, \dots, \xi_p^j \frac{\partial}{\partial x_j})|_{e_2}
\end{aligned}$$

□

Proposition 5.9. \mathcal{E} is a quasi-isomorphism.

Proof. In [34, Theorem 4.10] the authors showed that $(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2}$ is quasi-isomorphic to the double complex $C^{q,p}(\wedge \mathfrak{g}_1^*, \otimes \mathcal{R}(G_2))$ (5.1). The later is also quasi-isomorphic to $C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*)$. Therefore \mathcal{E} as a chain map with one side inverse between two quasi-isomorphic complexes is a quasi-isomorphism. □

we can also give a direct proof here to show that $C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*)$ is quasi-isomorphic to $(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2}$.

Proof. we will take a look at the triple complex $C_{\mathcal{R}}^s(G_2, \Omega_{L_2}^t \otimes \wedge^q \mathfrak{g}_1^*)$. During the first step we will fix q and vary s, t and view it as a double complex for each fixed q . we put an augmented row

$$C_{\mathcal{R}}^s(G_2, \mathbb{C} \otimes \wedge^q \mathfrak{g}_1^*)$$

there is an inclusion

$$0 \rightarrow C_{\mathcal{R}}^s(G_2, \mathbb{C} \otimes \wedge^q \mathfrak{g}_1^*) \xrightarrow{\varepsilon_1} C_{\mathcal{R}}^s(G_2, \Omega_{L_2}^0 \otimes \wedge^q \mathfrak{g}_1^*) \rightarrow C_{\mathcal{R}}^s(G_2, \Omega_{L_2}^1 \otimes \wedge^q \mathfrak{g}_1^*) \rightarrow \dots$$

and an augmented column

$$(\Omega_{L_2}^t)^{G_2} \otimes \wedge^q \mathfrak{g}_1^*$$

there is another inclusion

$$0 \rightarrow (\Omega_{L_2}^t)^{G_2} \otimes \wedge^q \mathfrak{g}_1^* \xrightarrow{\varepsilon_2} C_{\mathcal{R}}^0(G_2, \Omega_{L_2}^t \otimes \wedge^q \mathfrak{g}_1^*) \rightarrow C_{\mathcal{R}}^1(G_2, \Omega_{L_2}^t \otimes \wedge^q \mathfrak{g}_1^*) \rightarrow \dots$$

the columns are exact for $s \geq 0$ since L is smoothly contractible. the rows are exact for $t \geq 0$ because of the homotopy below (cf., [25, Lemma 2.3]):

$$\begin{aligned} H : C_{\mathcal{R}}^s(G_2, \Omega_{L_2}^t \otimes \wedge^q \mathfrak{g}_1^*) &\rightarrow C_{\mathcal{R}}^{s-1}(G_2, \Omega_{L_2}^t \otimes \wedge^q \mathfrak{g}_1^*) \\ H(c)(\psi_1, \dots, \psi_p) &= \pi(f_{(\psi_1, \dots, \psi_p)}^c) \end{aligned}$$

where

$$f_{(\psi_1, \dots, \psi_p)}^c(h) := c(hl, \psi_1, \dots, \psi_p)(l) \quad l \in L_2, h \in H_2$$

and π the projection to its H_2 invariant part, since H_2 is reductive and any H_2 module is completely reducible. By the standard diagram chasing of double complex we have that ε_1 and ε_2 both induces isomorphisms to the cohomology of the double complex. Therefore these augmented complexes are quasi-isomorphic, while the augmented column is isomorphic to the complex $(\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{l}_2)^*)^{\mathfrak{h}_2}$. □

5.3. Main theorem.

Lemma 5.10. *The map $\Phi_D \circ \Theta$ from $(\sum_{p+q=n} C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*), d_1)$ to the Hochschild complex of \mathcal{H} is a quasi-isomorphism.*

Proof. Recall from Corollary 3.23, we shall identify $\mathcal{H}_{(\delta, \sigma)}^{\mathfrak{h}}$ and $\text{Im}(\lambda^{\mathfrak{h}})$. In order to prove the statement we need to understand the Hochschild cohomology of the algebra \mathcal{A} with coefficients in the module \mathbb{C} given by the augmentation ε on \mathcal{A} .

We use the abstract version of the Hochschild-Serre spectral sequence.

The following is from [5, Thm 6.1, p. 349]

A subalgebra $\mathcal{A}_1 \subset \mathcal{A}$ of an augmented algebra \mathcal{A} is called normal if the left ideal J generated by $\text{Ker } \varepsilon_1$, where $\varepsilon_1 = \varepsilon|_{\mathcal{A}_1}$, is also a right ideal.

we let

$$(5.28) \quad \mathcal{A}_2 = \mathcal{A}/J, \quad \text{or equivalently} \quad \mathcal{A}_2 = \mathbb{C} \otimes_{\mathcal{A}_1} \mathcal{A},$$

and then we have a spectral sequence converging to the Hochschild cohomology $H_{\mathcal{A}}^*(\mathbb{C})$ of augmented algebra \mathcal{A} with coefficients in \mathbb{C} and with E_2 term given by

$$H_{\mathcal{A}_2}^p(H_{\mathcal{A}_1}^q(\mathbb{C})).$$

In our case we let $\mathcal{A} = C_c^\infty(G_1) \rtimes G_2^\delta$ and $\mathcal{A}_1 = C_c^\infty(G_1)$, with augmentation

$$\varepsilon(fU_\psi^*) = f(1) \quad \forall f \in C_c^\infty(G_1), \psi \in G_2.$$

Thus the ideal J is generated by

$$gU_\psi^*, \quad g(1) = 0.$$

It is a two sided ideal because $\forall \psi \in G_2, \tilde{\psi}(1) = 1$ and

$$g(1) = 0 \quad \text{iff} \quad (g \circ \tilde{\psi})(1) = 0$$

therefore the algebra $\mathcal{A}_2 = \mathbb{C} \otimes_{\mathcal{A}_1} \mathcal{A}$ is the group ring of G_2 .

Now since the original version of Hochschild-Serre spectral sequence is for unital algebra, in order to apply the spectral sequence we need one more preparation, i.e., we would like to extend the result to our algebra.

A algebra \mathcal{A} is said to have local units if for every finite family of elements $a_i \in \mathcal{A}$ there is an element $u \in \mathcal{A}$ such that $ua_i = a_iu = a_i$ for all i .

If an algebra has local units, it is excisive for Hochschild cohomology. [26, p32][40].

Now we let $\mathcal{A} = C_c^\infty(G_1) \rtimes G_2^\delta$ and $\mathcal{A}_1 = C_c^\infty(G_1)$. It is obvious that both our algebras \mathcal{A}_1 and \mathcal{A} have local units (just take $\text{Id}|_{\bigcup \text{supp}(f_i)} U_e^*$ and $\text{Id}|_{\bigcup \text{supp}(f_i)}$). Hence we add a unit to \mathcal{A}_1 as usual:

$$0 \rightarrow \mathbb{C} \rightarrow \tilde{\mathcal{A}}_1 \rightarrow \mathcal{A}_1 \rightarrow 0$$

where $\tilde{\mathcal{A}}_1 = \mathbb{C} \oplus \mathcal{A}_1$ with multiplication $(\lambda, u)(\mu, v) = (\lambda\mu, \lambda v + u\mu + uv)$ and unit $(1, 0)$, and add a unit to \mathcal{A} :

$$0 \rightarrow \mathbb{C}\mathcal{A}_2 \rightarrow \tilde{\mathcal{A}} \rightarrow \mathcal{A} \rightarrow 0$$

where $\tilde{\mathcal{A}} = \mathbb{C}\mathcal{A}_2 \oplus \mathcal{A}$ with multiplication $(\lambda, u)(\mu, v) = (\lambda\mu, \lambda v + u\mu + uv)$ and unit $(1e, 0)$. Thus we can have a diagram as below:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_{\mathcal{A}_2}^p(H_{\mathcal{A}_1}^q(\mathbb{C})) & \longrightarrow & H_{\mathcal{A}_2}^p(H_{\tilde{\mathcal{A}}_1}^q(\mathbb{C})) & \longrightarrow & H_{\mathcal{A}_2}^p(H_{\mathbb{C}}^q(\mathbb{C})) \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & H_{\mathcal{A}}^{p+q}(\mathbb{C}) & \longrightarrow & H_{\tilde{\mathcal{A}}}^{p+q}(\mathbb{C}) & \longrightarrow & H_{\mathcal{A}_2}^{p+q}(\mathbb{C}) \longrightarrow \dots \end{array}$$

the first row is due to the excision property of \mathcal{A}_1 and the second row is due to the excision property of \mathcal{A} . Now we apply the abstract Hochschild-Serre spectral sequence for the middle and right columns and conclude that $H_{\mathcal{A}_2}^p(H_{\mathcal{A}_1}^q(\mathbb{C}))$ also converge to $H_{\mathcal{A}}^{p+q}(\mathbb{C})$. This proved the Hochschild-Serre spectral sequence for our algebra \mathcal{A} .

Now $\text{Im}(\lambda^\natural)$ can be teated as \mathcal{A} with the weak topology given by the range of λ , we can have a corresponding spectral sequence which converges to the Hochschild cohomology of \mathcal{H} and whose E_2 term is given by the representative group cohomology of G_2 with coefficients in $H_{\mathcal{A}_1}^*(\mathbb{C})$, which is given by $\wedge^q \mathfrak{g}_1^*$. On E_2 level, the map $\Phi_D \circ \Theta$ realize the representative group cochains of G_2 with coefficients in $\wedge^q \mathfrak{g}_1^*$.

□

We now have the main result:

Theorem 5.11. *Let (G_1, G_2) be a matched pair of Lie groups, and L_2 be a nucleus of G_2 that is nilpotent. Let $\mathfrak{h}_2, \mathfrak{g}_1, \mathfrak{g}_2$ and \mathfrak{l}_2 denote the Lie algebras of $H_2 := G_2/L_2, G_1, G_2$ and L_2 respectively. Let us also assume that H_2 is G_1 -invariant, the action \triangleright of \mathfrak{h}_2 on \mathfrak{g}_1 is given by derivations, and the right G_1 action on L_2 is affine in its exponential coordinates. Then we have $\Phi_D \circ \mathcal{D}$ induces an isomorphism from the relative Lie cohomology of the pair $(\mathfrak{g}_1 \rtimes \mathfrak{g}_2, \mathfrak{h}_2)$ to the periodic cyclic cohomology of $\mathcal{H} = (\mathcal{U}(\mathfrak{g}_1) \rtimes \mathcal{R}(G_2))^{\text{cop}}$ with coefficient ${}^\sigma \mathbb{C}_\delta$, shifted by $\dim(G_1)$. We have*

$$(5.29) \quad \begin{aligned} \bigoplus_{i \text{ even}} H^i(\mathfrak{g}_1 \rtimes \mathfrak{g}_2, \mathfrak{h}_2, \mathbb{C}) &\cong HP^{\text{even}}(\mathcal{H}, {}^\sigma \mathbb{C}_\delta) \\ \bigoplus_{i \text{ odd}} H^i(\mathfrak{g}_1 \rtimes \mathfrak{g}_2, \mathfrak{h}_2, \mathbb{C}) &\cong HP^{\text{odd}}(\mathcal{H}, {}^\sigma \mathbb{C}_\delta) \end{aligned}$$

if $\dim(G_1)$ is even, and

$$(5.30) \quad \bigoplus_{i \text{ even}} H^i(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{h}_2, \mathbb{C}) \cong HP^{odd}(\mathcal{H}, {}^\sigma \mathbb{C}_\delta)$$

$$\bigoplus_{i \text{ odd}} H^i(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{h}_2, \mathbb{C}) \cong HP^{even}(\mathcal{H}, {}^\sigma \mathbb{C}_\delta)$$

if $\dim(G_1)$ is odd.

Proof. $\Phi_D \circ \Theta$ is a map between complexes $\{\sum C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*), d_1, d_2\}$ and $\{CC^\bullet(\mathcal{H}; {}^\sigma \mathbb{C}_\delta), b, B\}$. By Lemma 5.10, The map $\Phi_D \circ \Theta$ induces an isomorphism on Hochschild cohomology, therefore it also induces an isomorphism on cyclic cohomology. By Proposition 5.9, \mathcal{E} induces isomorphism between relative Lie cohomology of the pair $(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{h}_2)$ and the cohomology of $(\sum_{p+q=n} C_{\mathcal{R}}^p(G_2, \wedge^q \mathfrak{g}_1^*), d_1, d_2)$. Therefore the map $\Phi_D \circ \mathcal{D} = \Phi_D \circ \Theta \circ \mathcal{E}$ induces an isomorphism from the relative Lie cohomology of the pair $(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{h}_2)$ to the periodic cyclic cohomology of $\mathcal{H} = (\mathcal{U}(\mathfrak{g}_1) \blacktriangleright \mathcal{R}(G_2))^{\text{cop}}$ with coefficient ${}^\sigma \mathbb{C}_\delta$. The shift by the number $\dim(G_1)$ is included in the formulation of Φ_D map. \square

6. EXAMPLE CONSTRUCTION AND CALCULATION

6.1. Real diamond group $\mathbb{R} \ltimes H_3$. The so called real diamond Lie algebra is the 4-dimensional solvable Lie algebra \mathfrak{d} with basis T, X, Y, Z satisfying the following commutation relations, see [3, 36]:

$$[T, X] = -Y, [T, Y] = X, [X, Y] = Z,$$

this real diamond Lie algebra $\mathfrak{d} = \mathbb{R} \ltimes \mathfrak{h}_3$ is an extension of the one-dimensional Lie algebra $\mathbb{R}T$ by the Heisenberg algebra \mathfrak{h}_3 with basis X, Y, Z .

We look at the corresponding Lie group $\mathbb{R} \ltimes H_3$. A group element is represented as $(\theta, x + \mathbf{i}y, z)$, while the group multiplication is given by :

$$(\theta_1, x_1 + \mathbf{i}y_1, z_1) \cdot (\theta_2, x_2 + \mathbf{i}y_2, z_2) =$$

$$(\theta_1 + \theta_2, e^{-\mathbf{i}\theta_2}(x_1 + \mathbf{i}y_1) + (x_2 + \mathbf{i}y_2), z_1 + z_2 - \frac{1}{2}Im((x_1 + \mathbf{i}y_1)e^{-\mathbf{i}\theta_2}(x_2 - \mathbf{i}y_2)))$$

and inverse is given by

$$(6.1) \quad (\theta, x + \mathbf{i}y, z)^{-1} = (-\theta, -e^{\mathbf{i}\theta}(x + \mathbf{i}y), -z)$$

when $\theta = 0$ we have the usual H_3 . Denote $G_1 = (\theta, 0, 0)$, $G_2 = (0, x + \mathbf{i}y, z)$, we have

$$(6.2) \quad (\theta, x + \mathbf{i}y, z) = (\theta, 0, 0) \cdot (0, x + \mathbf{i}y, z), \quad \text{i.e., } G = G_1 G_2.$$

Because

$$(6.3) \quad (0, x + \mathbf{i}y, z_1) \cdot (\theta, 0, 0) = (\theta, e^{-\mathbf{i}\theta}(x + \mathbf{i}y), z) = (\theta, 0, 0) \cdot (0, e^{-\mathbf{i}\theta}(x + \mathbf{i}y), z)$$

we have

$$(6.4) \quad G_2 \triangleright G_1 : (0, x + \mathbf{i}y, z) \triangleright (\theta, 0, 0) = (\theta, 0, 0),$$

$$G_2 \triangleleft G_1 : (0, x + \mathbf{i}y, z) \triangleleft (\theta, 0, 0) = (0, e^{-\mathbf{i}\theta}(x + \mathbf{i}y), z)$$

Now we look at the Lie algebra:

$$\mathfrak{d} = \{T, X, Y, Z\} = \{e_1, e_2, e_3, e_4\} \quad \mathfrak{g}_2 = \{X, Y, Z\} \quad \mathfrak{g}_1 = \{T\}.$$

with Lie bracket given by

$$[T, X] = -Y, [T, Y] = X, [X, Y] = Z$$

and actions:

$$(6.5) \quad \mathfrak{g}_2 \triangleright \mathfrak{g}_1 : \text{trivial action} \quad \mathfrak{g}_2 \triangleleft \mathfrak{g}_1 : X \triangleleft T = Y, Y \triangleleft T = -X.$$

The cohomology ring of $H^*(\mathfrak{d})$ is generated by θ_1 , $\theta_2 \wedge \theta_3 \wedge \theta_4$ and $\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4$: H^0 is one dimensional, generated by $\{1\}$, H^1 is one dimensional, generated by $\{\theta_1\}$, H^3 is one dimensional, generated by $\{\theta_2 \wedge \theta_3 \wedge \theta_4\}$, H^4 is one dimensional, generated by $\{\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4\}$.

On G_1 we let $\theta_1 = d\theta$.

On G_2 we use the global exponential coordinate system $\begin{bmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ of G_2 to write

$$\theta_2 = dx, \theta_3 = dy, \theta_4 = \frac{y}{2}dx - \frac{x}{2}dy + dz.$$

Let $\varphi = (\theta, 0, 0)$, $\psi = (0, x + iy, z)$, we calculate

$$(6.6) \quad \nu(\varphi, \psi) = \varphi(\psi \triangleleft \varphi)^{-1} = (\theta, -e^{-i\theta}(x + iy), -z)$$

$$(6.7) \quad = (\theta, (-\cos \theta x - \sin \theta y) + i(-\cos \theta y + \sin \theta x), -z)$$

Now use 5.23 we calculate

$$\nu^*(\theta_1) = d\theta, \quad \text{therefore,} \quad \mu(\theta_1) = 1 \otimes \theta_1$$

$$\begin{aligned} \nu^*(\theta_2 \wedge \theta_3 \wedge \theta_4) &= \nu^*(dx \wedge dy \wedge dz) \\ &= d(-\cos \theta x - \sin \theta y) \wedge d(-\cos \theta y + \sin \theta x) \wedge dz \\ &= (\cos^2 \theta + \sin^2 \theta) dx \wedge dy \wedge dz \\ &\quad + (\sin^2 \theta x - \sin \theta \cos \theta y + \cos \theta \sin \theta y + \cos^2 \theta x) d\theta \wedge dx \wedge dz \\ &\quad + (-\sin \theta \cos \theta x + \cos^2 \theta y + \sin^2 \theta y + \cos \theta \sin \theta x) d\theta \wedge dy \wedge dz \\ &= dx \wedge dy \wedge dz + x d\theta \wedge dx \wedge dz + y d\theta \wedge dy \wedge dz \end{aligned}$$

hence

$$\begin{aligned} \mu_0(\theta_2 \wedge \theta_3 \wedge \theta_4) &= dx \wedge dy \wedge dz \otimes 1 \\ \mu_1(\theta_2 \wedge \theta_3 \wedge \theta_4) &= (x dx \wedge dz + y dy \wedge dz) \otimes \theta_1 \end{aligned}$$

$$\begin{aligned} \nu^*(\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4) &= \nu^*(d\theta \wedge dx \wedge dy \wedge dz) \\ &= d\theta \wedge d(-\cos \theta x - \sin \theta y) \wedge d(-\cos \theta y + \sin \theta x) \wedge dz \\ &= (\cos^2 \theta + \sin^2 \theta) d\theta \wedge dx \wedge dy \wedge dz \end{aligned}$$

hence

$$\mu(\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4) = d\theta \wedge dx \wedge dy \wedge dz \otimes 1$$

Therefore the images of θ_1 and $\theta_2 \wedge \theta_3 \wedge \theta_4$ give two even cocycles and the image of $1, \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4$ give two odd cocycles. We take $\theta_2 \wedge \theta_3 \wedge \theta_4$ as an example.

$$\mathcal{E}(\theta_2 \wedge \theta_3 \wedge \theta_4)(\psi_0, \dots, \psi_3) = 1 \cdot \int_{\Delta(\psi_0, \dots, \psi_3)} dx \wedge dy \wedge dz$$

where

$$\begin{aligned} dx &= (x_1 - x_0)dt_1 + (x_2 - x_0)dt_2 + (x_3 - x_0)dt_3, \\ dy &= (y_1 - y_0)dt_1 + (y_2 - y_0)dt_2 + (y_3 - y_0)dt_3, \\ dz &= (z_1 - z_0)dt_1 + (z_2 - z_0)dt_2 + (z_3 - z_0)dt_3. \end{aligned}$$

therefore the $D^{3,0}$ part of $\theta_2 \wedge \theta_3 \wedge \theta_4$ is

$$\mathcal{E}(\theta_2 \wedge \theta_3 \wedge \theta_4)(\psi_0, \dots, \psi_3) = \frac{1}{2} \sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)} - x_0)(y_{\sigma(2)} - y_0)(z_{\sigma(3)} - z_0)$$

Now for the other component we have

$$\mathcal{E}(\theta_2 \wedge \theta_3 \wedge \theta_4)(\psi_0, \dots, \psi_2) = \theta_1 \cdot \int_{\Delta(\psi_0, \dots, \psi_2)} y \cdot dy \wedge dz x \cdot dx \wedge dz$$

where

$$\begin{aligned} dx &= (x_1 - x_0)dt_1 + (x_2 - x_0)dt_2, \\ dy &= (y_1 - y_0)dt_1 + (y_2 - y_0)dt_2, \\ dz &= (z_1 - z_0)dt_1 + (z_2 - z_0)dt_2. \end{aligned}$$

therefore the $D^{2,1}$ part of $\theta_2 \wedge \theta_3 \wedge \theta_4$ is

$$\begin{aligned} &\mathcal{E}(\theta_2 \wedge \theta_3 \wedge \theta_4)(\psi_0, \dots, \psi_2) \\ &= \frac{1}{6} \theta_1 \cdot ((y_0^2 + x_0^2)(z_1 - z_2) + (y_1^2 + x_1^2)(z_2 - z_0) + (y_2^2 + x_2^2)(z_0 - z_1)) \end{aligned}$$

Use Φ to go direct from these two parts to the representative cochain on algebra $\mathcal{A} = C_c^\infty(G_1) \rtimes G_2^\delta$, we have $\Phi(\theta_2 \wedge \theta_3 \wedge \theta_4) \in C^4(\mathcal{A}) \cup C^2(\mathcal{A})$.

As an example, the $C^2(\mathcal{A})$ part is:

$$\begin{aligned} &\Phi(\theta_2 \wedge \theta_3 \wedge \theta_4)(f_0 U_{\psi_0}^*, f_1 U_{\psi_1}^*, f_2 U_{\psi_2}^*) \\ &= ((y_0^2 + x_0^2)(z_1 - z_2) + (y_1^2 + x_1^2)(z_2 - z_0) + (y_2^2 + x_2^2)(z_0 - z_1)) \int_{\mathbb{R}} f_0 f_1 f_2 \theta_1 \end{aligned}$$

6.2. Other examples. Other examples like higher diamond group can also be calculated explicitly. We can use mathematical software like MapleTM to initiate example Lie groups and Lie algebras and calculate all the Lie algebra cohomology classes we want, and then implement \mathcal{E} , Θ and Φ maps as in the paper.

7. APPENDIX

7.1. detail of step 1. We give a detailed account of the transition from $CC^\bullet(\mathcal{R}(G_2)) \blacktriangleright \triangleleft \mathcal{U}(\mathfrak{g}_1), \sigma^{-1} \mathbb{C}_\delta$ to $C^{\bullet, \bullet}(\wedge \mathfrak{g}_1^*, \otimes \mathcal{R}(G_2))$ in step 1 in [subsection 5.1](#). We just restate what is

proved in [34, 4.2], under our setting.

Define a bi-cyclic complex $C^{\bullet,\bullet}(\mathcal{U}, \mathcal{R}, \sigma^{-1}\mathbb{C}_\delta)$, where

$$C^{p,q}(\mathcal{U}, \mathcal{R}, \sigma^{-1}\mathbb{C}_\delta) := \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{R}^p \otimes \mathcal{U}^q, \quad p, q \geq 0$$

$$(7.1) \quad \begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \uparrow b_{\mathcal{U}} & \downarrow B_{\mathcal{U}} & \uparrow b_{\mathcal{U}} & \downarrow B_{\mathcal{U}} & \uparrow b_{\mathcal{U}} & \downarrow B_{\mathcal{U}} \\ \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{U}^{\otimes 2} & \xrightleftharpoons[b_{\mathcal{R}}]{b_{\mathcal{R}}} & \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{U}^{\otimes 2} \otimes \mathcal{R} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{U}^{\otimes 2} \otimes \mathcal{R}^{\otimes 2} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} \dots \\ \uparrow b_{\mathcal{U}} & \downarrow B_{\mathcal{U}} & \uparrow b_{\mathcal{U}} & \downarrow B_{\mathcal{U}} & \uparrow b_{\mathcal{U}} & \downarrow B_{\mathcal{U}} \\ \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{U} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{U} \otimes \mathcal{R} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{U} \otimes \mathcal{R}^{\otimes 2} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} \dots \\ \uparrow b_{\mathcal{U}} & \downarrow B_{\mathcal{U}} & \uparrow b_{\mathcal{U}} & \downarrow B_{\mathcal{U}} & \uparrow b_{\mathcal{U}} & \downarrow B_{\mathcal{U}} \\ \sigma^{-1}\mathbb{C}_\delta & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{R} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} & \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{R}^{\otimes 2} & \xrightleftharpoons[B_{\mathcal{R}}]{b_{\mathcal{R}}} \dots \end{array}$$

We refer to [31, Sec. 3.2] for the detail of the construction of the above complex and the proof of its bi-cyclicity.

Next, we identify the diagonal of the above bi-cyclic complex $D^\bullet(\mathcal{U}, \mathcal{R}, \sigma^{-1}\mathbb{C}_\delta)$ with the standard Hopf cyclic module $CC^\bullet(\mathcal{R}(G_2) \blacktriangleleft \mathcal{U}(\mathfrak{g}_1), \sigma^{-1}\mathbb{C}_\delta)$ by the map, which is similar to the one defined in [31, p. 766-767] and [34, p. 502], $\Psi_{\blacktriangleleft} : CC^\bullet(\mathcal{R}(G_2) \blacktriangleleft \mathcal{U}(\mathfrak{g}_1), \sigma^{-1}\mathbb{C}_\delta) \rightarrow D^\bullet(\mathcal{U}, \mathcal{R}, \sigma^{-1}\mathbb{C}_\delta)$:

$$\begin{aligned} & \Psi_{\blacktriangleleft}(1 \otimes F^1 \blacktriangleleft u^1 \otimes \dots \otimes F^n \blacktriangleleft u^n) \\ &= 1 \otimes F^1 \otimes F^2 S(u_{<n-1>}^1) \otimes F^3 S(u_{<n-2>}^1) S(u_{<n-2>}^2) \otimes \dots \\ & \quad \otimes F^n S(u_{<1>}^1) S(u_{<1>}^2) \dots S(u_{<1>}^{n-1}) \otimes u_{<0>}^1 \otimes \dots \otimes u_{<0>}^{n-1} \otimes u^n, \\ & \Psi_{\blacktriangleleft}^{-1}(1 \otimes F^1 \otimes \dots \otimes F^n \otimes u^1 \otimes \dots \otimes u^n) \\ &= 1 \otimes F^1 \blacktriangleleft u_{<0>}^1 \otimes F^2 u_{<1>}^1 \blacktriangleleft u_{<0>}^2 \otimes \dots \otimes F^n u_{<n-1>}^1 \dots u_{<1>}^{n-1} \blacktriangleleft u^n. \end{aligned}$$

We relate $\text{Tot}C^{\bullet,\bullet}(\mathcal{U}, \mathcal{R}, \sigma^{-1}\mathbb{C}_\delta)$ with the diagonal $D^\bullet(\mathcal{U}, \mathcal{R}, \sigma^{-1}\mathbb{C}_\delta)$ by Alexander-Whitney map and shuffle map (cf. [23]):

$$(7.2) \quad \begin{aligned} AW &= \bigoplus_{p+q=n} AW_{p,q}, \\ AW_{p,q} : \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{U}^q \otimes \mathcal{R}^p &\rightarrow \sigma^{-1}\mathbb{C}_\delta \otimes \mathcal{U}^{p+q} \otimes \mathcal{R}^{p+q} \\ AW_{p,q} &= (-1)^{p+q} \uparrow \partial_0 \dots \uparrow \partial_0 \overrightarrow{\partial}_{p+q} \dots \overrightarrow{\partial}_{p+1}, \\ Sh &= \sum_{\sigma \in Sh(p,q)} (-1)^\sigma \uparrow s_{\sigma(1)} \dots \uparrow s_{\sigma(p)} \otimes \overrightarrow{s}_{\sigma(p+1)} \dots \overrightarrow{s}_{\sigma(p+q)}. \end{aligned}$$

We note that both Alexander-Whitney map and shuffle map are only chain maps between Hochschild complexes. The lack of explicit cyclic Alexander-Whitney map is the reason that the vertical map in diagram 1.1 can not be inverted.

Follow the treatment in [34, Sec. 4.2], we can go from the complex $C^{p,q}(\mathcal{U}, \mathcal{R}, \sigma^{-1}\mathbb{C}_\delta)$ to the quasi-isomorphic complex $C^{p,q}(\wedge \mathfrak{g}_1, \otimes \mathcal{R}(G_2), \sigma^{-1}\mathbb{C}_\delta) := \sigma^{-1}\mathbb{C}_\delta \otimes \wedge^q \mathfrak{g}_1 \otimes \mathcal{R}(G_2)^{\otimes p}$ through antisymmetrization map. We can continue from the complex $C^{\bullet,\bullet}(\wedge \mathfrak{g}_1, \otimes \mathcal{R}(G_2), \sigma^{-1}\mathbb{C}_\delta)$ to the quasi-isomorphic complex $C^{\bullet,\bullet}(\wedge \mathfrak{g}_1^*, \otimes \mathcal{R}(G_2)) := \wedge^\bullet \mathfrak{g}_1^* \otimes \mathcal{R}(G_2)^{\otimes \bullet}$ by Poincaré isomorphism:

$$(7.3) \quad \begin{aligned} \mathfrak{D} : \wedge^{\dim(\mathfrak{g}_1)} \mathfrak{g}_1^* \otimes \wedge^{\dim(\mathfrak{g}_1)-q} \mathfrak{g}_1 \otimes \mathcal{R}(G_2)^{\otimes p} &\rightarrow \wedge^q \mathfrak{g}_1^* \otimes \mathcal{R}(G_2)^{\otimes p} \\ \mathfrak{D}(\varpi \otimes \eta \otimes F^1 \otimes \cdots \otimes F^p) &= \iota(\eta)\varpi \otimes F^1 \otimes \cdots \otimes F^p, \end{aligned}$$

where ϖ is a volume form and $\iota(\eta)$ is the contraction by η .

As in [34, Sec. 4.2] and [32, Sec. 1.3], we have the right coadjoint action of \mathfrak{g}_1 on ϖ

$$(7.4) \quad \text{ad}^*(Z)\varpi = \delta(Z)\varpi, \quad \forall Z \in \mathfrak{g}_1,$$

and identify $\wedge^{\dim(\mathfrak{g}_1)} \mathfrak{g}_1^*$ with \mathbb{C}_δ as right \mathfrak{g}_1 -modules.

use 3.16 and 3.39 we have the left coaction of $\mathcal{R}(G_2)$ on ϖ^*

$$(7.5) \quad \blacktriangledown(\varpi) = \sigma^{-1} \otimes \varpi,$$

and identify $\wedge^{\dim(\mathfrak{g}_1)} \mathfrak{g}_1^*$ with $\sigma^{-1}\mathbb{C}_\delta$ as right \mathfrak{g}_1 -module and left $\mathcal{R}(G_2)$ -comodule, or, at the same time, as right-left SAYD over $\mathcal{R}(G_2) \blacktriangleright \blacktriangleleft \mathcal{U}(\mathfrak{g}_1)$.

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